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Homogenní prostory

Katedra matematické analýzy

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Prohlašuji, že jsem svou bakalářskou práci napsala samostatně a výhradně s použitím citovaných pramenů. Souhlasím se zapůjčováním práce.

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Abstrakt: Sobolevovy prostory hrají fundamentální roli v moderní teorii PDR. Jejich nejdůležitější charakteristikou jsou vnoření do dalších prostorů funkcí, jelikož povaha těchto vnoření určuje mnohé vlastnosti úloh v PDR formulovaných na příslušném Sobolevově prostoru. Jeden z aspektů vnoření Sobolevova prostoru do Lebesgueova prostoru souvisí s otázkou platnosti Poincarého nerovnosti na uvažované oblasti. To se stalo podnětem pro W. D. Evanse a D. J. Harrise ke zkoumání Poincarého nerovnosti na oblastech se zobecněnou páteří, jehož závěr byl prezentován v [2]. Vzhledem k nesmírnému významu Sobolevových prostorů se poslední dobou věnuje velká pozornost jejich rozšíření do prostředí metrických prostorů s mírou. Naším cílem je představit tři taková zobecnění a dokázat větu v duchu výsledků W. D. Evanse a D. J. Harrise pro metrické prostory.

Klíčová slova: Sobolevovy prostory, Poincarého nerovnost, metrický prostor s mírou, prostory homogenního typu, horní gradient

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Abstract: Sobolev spaces play a fundamental role in the modern theory of PDEs. Their most important characteristics are embeddings into other function spaces, since the nature of these embeddings determines many of the properties of problems in PDEs formulated on the corresponding Sobolev space. One of the features of an embedding of a Sobolev space into a Lebesgue space is associated with the question of the validity of the Poincaré inequality on a domain under consideration. This became a motivation of W. D. Evans and D. J. Harris for investigation of the Poincaré inequality on generalized ridged domains, outcome of which was presented in [2]. Because of the significance of Sobolev spaces, much attention is paid recently to their extensions to the setting of metric measure spaces. Our aim is to introduce three of such generalizations and to establish a theorem in the spirit of the results due to W. D. Evans and D. J. Harris in the environment of metric spaces.

Keywords: Sobolev spaces, Poincaré inequality, metric measure space, spaces of homogeneous type, upper gradient

Introduction

This thesis is organized into three chapters.

The first chapter is devoted to the classical theory of Sobolev spaces on domains in $\mathbb{R}^n$ with focus on their embedding properties according to the geometrical features of the domain. It also involves some elementary facts from real and functional analysis as a necessary background. All the proofs are omitted and can be found mainly in the monograph of R. A. Adams [1], which is the major source of information given in this section. At the end of the chapter, we cite the results of W. D. Evans and D. J. Harris from [2], the point of departure of our research.

The second chapter contains a unified survey of the theory of Sobolev spaces on metric measure spaces. Of course it is by no means complete, we just describe three different approaches to the Sobolev-type spaces and discuss their mutual relationship. For details we refer the reader to [3] and [4]. This is preceded by a short introduction to the theory of metric measure spaces, including doubling measures and maximal functions, according to [5]. Again, we do not bring in any proofs. In case of further interest, see materials to which we refer here.

In the third chapter, as its name itself prompts, we present our principal theorem. Roughly speaking, it deals with a relationship between the Poincaré inequality on an open set in a metric measure space and the Poincaré inequality on an interval with appropriately chosen measure according to this open set. We present a complete proof of this result.

Chapter 1

Sobolev spaces on $\mathbb{R}^n$

To begin, we would like to briefly introduce Sobolev spaces, to whose development a Russian mathematician S.L. Sobolev most considerably contributed. Nowadays, these spaces constitute a very significant structure in functional analysis, as they play the fundamental role in the modern theory of partial differential equations. Nevertheless, the scope of their applications is not limited only to this branch of analysis, but includes also algebraic topology, complex analysis, differential geometry and probability theory. Numerous generalizations and extensions of Sobolev spaces have recently been established and they are still widely studied.

First, let us recall some necessary terms.

Definition 1.0.1 An open connected set $\Omega \subset X$, where (X, ρ) is a metric space, is called a *domain*.

Definition 1.0.2 An arranged n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ of nonnegative integers α_i is called *multiindex* and quantity $|\alpha| = \sum_{i=1}^{n} \alpha_i$ is called the *height of multiindex* α.

Definition 1.0.3 Let $\Omega \subset \mathbb{R}^n$ be a domain, $k \in \mathbb{N}$ and $0 < \lambda \leq 1$. We define:

- $C^k(\Omega)$ as the linear space of all functions on Ω, which have continuous all partial derivatives of orders $0 \leq |\alpha| \leq k$ on Ω,
- $C_0^\infty(\Omega) = \{f \in \bigcap_{k=0}^{\infty} C^k(\Omega) : \text{supp } f \subset \Omega \text{ is compact}\}$, where $\text{supp } f = \overline{\{x \in \Omega : f(x) \neq 0\}}$,
- $C_B^k(\Omega) = \{f \in C^k(\Omega) : D^\alpha f \text{ is bounded on } \Omega \text{ for } |\alpha| \leq j\}$, equipped with the norm $\|f\|_{C_B^k(\Omega)} = \max\limits_{0 \leq |\alpha| \leq k} \sup\limits_{x \in \Omega} |D^\alpha f(x)|$,
- $C^k(\overline{\Omega}) = \{f \in C^k(\Omega) : D^\alpha f \text{ is bounded and uniformly continuous on } \Omega \text{ for all } 0 \leq |\alpha| \leq k\}$, equipped with the norm $\|f\|_{C^k(\overline{\Omega})} = \max\limits_{0 \leq |\alpha| \leq k} \sup\limits_{x \in \Omega} |D^\alpha f(x)|$,
- $C^{k,\lambda}(\overline{\Omega}) = \{f \in C^k(\overline{\Omega}) : \text{there exists a constant } K > 0 \text{ such that } |D^\alpha f(x) - D^\alpha f(y)| \leq K|x-y|^\lambda \text{ for all } 0 \leq |\alpha| \leq k \text{ and } x, y \in \Omega\}$, equipped with the norm $\|f\|_{C^{k,\lambda}(\overline{\Omega})} = \|f\|_{C^k(\overline{\Omega})} + \max\limits_{0 \leq |\alpha| \leq k} \sup\limits_{x,y \in \Omega; x \neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\lambda}$.

Definition 1.0.4 Let (X, S, μ) be a measure space. For a μ-measurable function f on X and $1 \leq p \leq \infty$ we establish the functional $\|\cdot\|_p$:

$$\|f\|_p = \begin{cases} \Big(\int\limits_X |f|^p d\mu\Big)^{1/p} & \text{if } 1 \leq p < \infty; \\ \operatorname{ess\,sup}\limits_{x\in X} |f(x)| & \text{if } p = \infty. \end{cases}$$

Next, we define a linear space $\mathcal{L}^p(X, S, \mu) = \mathcal{L}^p(X) = \{f\ \mu\text{-measurable on } X : \|f\|_p < \infty\}$ and an equivalence relation on this space: $f \sim g$ if $f = g$ μ-a.e.
Finally, we get to the definition of a normed linear space $L^p(X, S, \mu) = L^p(X)$[1] as the quotient $\mathcal{L}^p(X)/\sim = \{[f] : f \in \mathcal{L}^p(X)\}$, where $[f] = \{g \in \mathcal{L}^p(X) : g \sim f\}$, endowed with norm $\|[f]\|_p := \|f\|_p$.

If not otherwise stated, in this chapter we deal with spaces $\mathbb{R}^n$ equipped with the n - dimensional Lebesgue measure λ_n. We set $\int_\Omega f(x)dx = \int_\Omega f(x)d\lambda_n(x) = \int_\Omega f d\lambda_n$ for $\Omega \subset \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ Lebesgue measurable[2].

Definition 1.0.5 Let $\Omega \subset \mathbb{R}^n$ be an arbitrary domain, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n$ a multiindex and $f, g_\alpha \in L^1_{\text{loc}}(\Omega)$. We say, that g_α is the *weak* or *distributional derivative* of f with respect to $x^\alpha = (x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots, x_n^{\alpha_n})$ (notation: $D^\alpha f = g_\alpha$), provided that

$$(-1)^{|\alpha|} \int_\Omega f(x) D^\alpha \Phi(x) dx = \int_\Omega g_\alpha(x)\Phi(x)dx$$

for every $\Phi \in C_0^\infty(\Omega)$.

Remark 1.0.6 Weak derivative is unique up to sets of measure zero[3]. If f is smooth enough, then the distributional derivative coincides with the classical one.

Now we are in the position to introduce Sobolev spaces of integer order over an arbitrary domain in $\mathbb{R}^n$.

Definition 1.0.7 For $k \in \mathbb{N}_0, 1 \leq p \leq \infty$ and domain $\Omega \subset \mathbb{R}^n$ we define *Sobolev space* over Ω, $W^{k,p}(\Omega)$, as follows:

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega),\ \forall \alpha\, \text{multiindex} : 0 \leq |\alpha| \leq k\},$$

where $D^\alpha f$ is a weak derivative of f. The elements of $W^{k,p}(\Omega)$ are equivalence classes of functions, two functions being equivalent if they are equal a.e. in Ω.
Next define a functional $\|\cdot\|_{k,p,\Omega}$ on $W^{k,p}(\Omega)$ this way:

$$\|f\|_{k,p,\Omega} = \begin{cases} \Big(\sum\limits_{0\leq|\alpha|\leq k} \|D^\alpha f\|^p_{L^p(\Omega)}\Big)^{1/p} & \text{if } 1 \leq p < \infty; \\ \max\limits_{0\leq|\alpha|\leq k} \|D^\alpha f\|_{L^\infty(\Omega)} & \text{if } p = \infty. \end{cases}$$

[1] We say that f belongs to $L^p_{\text{loc}}(X)$ if $f \in L^p(A)$ for every $A \subset X$ such that $\overline{A}$ is a compact subset of X.

[2] Similarly, we identify $\int_\Omega f(x)d\mu(x) = \int_\Omega f d\mu$ for any measure space (X, S, μ) and $\Omega \subset X$ and $f : \Omega \to \mathbb{R}$ $\mu-$measurable.

[3] If some property holds on set $A \subset X$, where (X, S, μ) is a measure space, for which $\mu(X \setminus A) = 0$, then we say that it holds μ-a.e.

Theorem 1.0.8 *For each $k \in \mathbb{N}_0, 1 \le p \le \infty$, the functional $\|\cdot\|_{k,p,\Omega}$ is a norm on $W^{k,p}(\Omega)$ and $W^{k,p}(\Omega)$ endowed with this norm is a Banach space, which is separable for $1 \le p < \infty$ and reflexive for $1 < p < \infty$. In particular,$W^{k,2}(\Omega)$ is a Hilbert space.*

Remark 1.0.9 There exist two other function spaces, which equipped with the appropriate norm from the preceding definition are also called Sobolev spaces over Ω. Namely:

$$\begin{aligned} H^{k,p}(\Omega) &\equiv \text{ the completion of } \{f \in C^k(\Omega) : \|f\|_{k,p,\Omega} < \infty\} \\ &\qquad \text{with respect to the norm } \|\cdot\|_{k,p,\Omega}, \\ W_0^{k,p}(\Omega) &\equiv \text{ the closure of } C_0^\infty(\Omega) \text{ in the space } W^{k,p}(\Omega), \end{aligned}$$

where $k \in \mathbb{N}_0, 1 \le p \le \infty$ again. What is now surely expected, is some discussion of the relationship of just presented three spaces. Following claims can be proved:

$$H^{k,p}(\Omega) = W^{k,p}(\Omega) \text{ for every domain } \Omega,\ 1 \le p < \infty,$$

$$W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n) \text{ for } 1 \le p < \infty.$$

This is associated with the question of the density of smooth functions in Sobolev space (in the sense of its first definition), which will be touched later on. Unless otherwise stated, we consider $W^{k,p}(\Omega)$ when speaking about a Sobolev space over Ω.

Since Sobolev spaces are Banach spaces, the problem of the representation of their dual spaces is pertinent. For further studies of Sobolev spaces, the information about possibility of approximation by smooth functions can be very worthy. The reason is that many proofs can be carried out quite easily for smooth functions and then just by taking limits verified for functions from appropriate Sobolev space, whereas a direct proof may be much more difficult or even impossible. Also the existence of a bounded extension operator mapping Sobolev space $W^{k,p}(\Omega)$, where Ω is a proper subset of $\mathbb{R}^n$, to Sobolev space $W^{k,p}(\mathbb{R}^n)$ with preservation of the values of mapped function on Ω, has proved to be useful, because space $W^{k,p}(\Omega)$ inherits then some properties possessed by a target space $W^{k,p}(\mathbb{R}^n)$. In this paragraph we just wanted to point out interesting issues concerning the theory of Sobolev spaces, but we won't discuss them deeper, since they are not directly connected with our main objectives. What we shall focus on are the embedding properties of Sobolev spaces, their principal characteristics, thanks to which they became an essential tool in the study of differential and integral operators.

Definition 1.0.10 A Linear operator $T : (X, \|\cdot\|_x) \to (Y, \|\cdot\|_y)$, $(X, \|\cdot\|_x)$ and $(Y, \|\cdot\|_y)$ being normed linear spaces, is called:

- *continuous* ($\equiv$ bounded), provided that there exists a constant $C > 0$ such that $\|T(x)\|_y \le C\|x\|_x$ holds for all $x \in X$,
- *compact*, provided that $\overline{T(\{x \in X : \|x\|_x \le 1\})}$ is a compact set in Y.

Definition 1.0.11

- We say that the normed linear space $(X, \|\cdot\|_x)$ is *continuously embedded* to the normed linear space $(Y, \|\cdot\|_y)$ (notation: $X \hookrightarrow Y$), if X is a vector subspace of Y and the identity operator $I : X \to Y$, $I(x) = x$ for all $x \in X$, is continuous.

- We say that the normed linear space $(X, \|\cdot\|_x)$ is *compactly embedded* to the normed linear space $(Y, \|\cdot\|_y)$ (notation: $X \hookrightarrow\hookrightarrow Y$), if X is a vector subspace of Y and the identity operator $I : X \to Y$, $I(x) = x$ for all $x \in X$, is compact.

Remark 1.0.12 The condition that X is a subspace of Y and I is an identity may be replaced by the weaker requirement of existence of certain canonical linear transformation of X into Y.

As for the target space, we distinguish five main types of embeddings of Sobolev spaces. They involve embeddings of $W^{k,p}(\Omega)$ to:

(i) $W^{m,q}(\Omega)$, in particular $L^q(\Omega)$, for $m \leq k, 1 \leq q \leq \infty$.

(ii) $C^m_B(\Omega)$ for $m \in \mathbb{N}$.

(iii) $C^{m,\lambda}(\overline{\Omega})$, in particular $C^m(\overline{\Omega})$, for $m \in \mathbb{N}$, $0 < \lambda \leq 1$.

(iv) $W^{m,q}(\Omega^d)$, in particular $L^q(\Omega^d)$, for $m \leq k, 1 \leq q \leq \infty, d < n$. Ω^d denotes the intersection of Ω with a d-dimensional plane in $\mathbb{R}^n$, considered as a domain in $\mathbb{R}^d$.

(v) $L^q(\partial\Omega)$ for $1 \leq q \leq \infty$. Here we assume that $\partial\Omega$ can be written as a union of $\{U_j\}$, where U_j is $n-1$-dimensional set in $\mathbb{R}^n$ having a parametrization. By integral over $\partial\Omega$ we intend then a sum over j of surface integrals over U_j.

The embedding of the first type is meant as the boundedness of the identity operator. But the interpretation of the embeddings of the other types is not so obvious, therefore it needs more detailed explanation. By the embedding of Sobolev space into any kind of the space of continuous functions, we understand that for each element of $W^{k,p}(\Omega)$, i.e. equivalence class of functions $[f]$, there exists a representative f_0 belonging to the target space (denote it $(C, \|\cdot\|_C)$ for this moment) and, furthermore, there is a constant $M > 0$ independent on $[f]$ such that $\|f_0\|_C \leq M\|f_0\|_{k,p,\Omega}$. In the cases (iv) and (v), the problem is how to define function on the set of measure zero in $\mathbb{R}^n$ as an image of the equivalence class consisting of functions defined and equal a.e. on $\Omega \subset \mathbb{R}^n$. The solution inheres in the density of smooth functions in $W^{k,p}(\Omega)$, which are already defined everywhere on Ω. In terms of (iv), we know that $W^{k,p}(\Omega) = H^{k,p}(\Omega)$, therefore after adding some more conditions, we can set the embedding image of $f \in W^{k,p}(\Omega)$ equal to the limit of traces of functions f_n on Ω^d in the appropriate target space, where f is the limit of a sequence f_n in $W^{k,p}(\Omega)$. To use the same method for the trace embedding on the boundary of Ω (type (v)), we need the dense subset to consist of functions continuously extendable to the $\partial\Omega$ (i.e. $C^m(\overline{\Omega})$), or the existence of the extension operator mapping $W^{k,p}(\Omega)$ into $W^{k,p}(\mathbb{R}^n)$. If the latter is satisfied, we apply the described technique on $C^\infty(\mathbb{R}^n)$ dense in $W^{k,p}(\mathbb{R}^n)$. Of course, this method can be used only under some circumstances. Except for the possibility of approximation by smooth functions, we have to guarantee the existence of that limit in target space. Trace embeddings on the boundary of domain are important for finding weak solutions of the problems formulated with the boundary conditions in PDEs.

The existence and quality of an embedding, as well as many other features of Sobolev spaces, depend on regularity properties of the domain, over which they are defined.

Such regularity or irregularity of a domain is usually expressed with some geometrical conditions, especially imposed on the boundary of a domain, that may or may not be satisfied by this domain. Naturally, the more restrictive requirements on the domain we have, the better embedding result we obtain.

Remark 1.0.13 If we consider Sobolev spaces $W_0^{k,p}(\Omega)$, the geometrical characteristics of the boundary of a domain don't play any role in its embedding properties. The reason is that an operator of extension by zero outside the domain Ω maps $W_0^{k,p}(\Omega)$ isometrically into $W^{k,p}(\mathbb{R}^n)$ for an arbitrary domain $\Omega \subset \mathbb{R}^n$.

Let's specify several important geometrical conditions that can be imposed on a domain. But first, we introduce some terms, which appear in the description of regularity properties of a domain. Majority of the following definitions have their origin in [1].

"Definition 1.0.14 Let Φ be a one-to-one transformation (bijection) of a domain $\Omega \subset \mathbb{R}^n$ onto a domain $G \subset \mathbb{R}^n$, having inverse $\Psi = \Phi^{-1}$. We call Φ *m-smooth* if the functions $\Phi_1, \ldots, \Phi_n$ belong to $C^m(\overline{\Omega})$ and the functions $\Psi_1, \ldots, \Psi_n$ belong to $C^m(\overline{G})$, where $y = \Phi(x) = (\Phi_1(x_1, \ldots, x_n), \ldots, \Phi_n(x_1, \ldots, x_n))$ and $x = \Psi(y) = (\Psi_1(y_1, \ldots, y_n), \ldots, \Psi_n(y_1, \ldots, y_n))$.

Definition 1.0.15 Given a point $x \in \mathbb{R}^n$, an open ball B_1 with center x, and an open ball B_2 not containing x, the set $C_x = B_1 \cap \{x + \lambda(y - x) : y \in B_2, \lambda > 0\}$ is called a *finite cone* in $\mathbb{R}^n$ having vertex at x.

Definition 1.0.16 An open cover $\mathcal{O}$ of a set $S \subset \mathbb{R}^n$ is said to be *locally finite* if any compact set in $\mathbb{R}^n$ can intersect at most finitely many elements of $\mathcal{O}$.

Remark 1.0.17 $\mathcal{O}$ must be countable. Furthermore, if S is closed, then any open cover of S possesses a locally finite subcover.

Definition 1.0.18 (I) Ω has the *cone property* if there exists a finite cone C such that each point $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω and congruent to C.

(II) Ω has the *strong local Lipschitz property* provided there exist positive numbers δ and M, a locally finite open cover $\{U_j\}$ of $\partial\Omega$, and for each U_j a real-valued function f_j of $n - 1$ real variables, such that the following conditions hold:

(i) For some $R \in \mathbb{N}$, every collection of $R + 1$ of the sets U_j has empty intersection.

(ii) For every pair of points $x, y \in \Omega_\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \delta\}$ such that $|x - y| < \delta$ there exists j such that

$$x, y \in \{x \in U_j : \mathrm{dist}(x, \partial U_j) > \delta\}.$$

(iii) Each function f_j satisfies a Lipschitz condition with constant M.

(iv) For some Cartesian coordinate system $(\xi_{j,1}, \ldots, \xi_{j,n})$ in U_j the set $\Omega \cap U_j$ is represented by the inequality

$$\xi_{j,n} < f_j(\xi_{j,1}, \ldots, \xi_{j,n-1}).$$

(III) Ω has the *uniform C^m-regularity property* if there exists a locally finite open cover $\{U_j\}$ of $\partial\Omega$, and a corresponding sequence $\{\Phi_j\}$ of m-smooth one-to-one transformations with Φ_j taking U_j onto $B = \{y \in \mathbb{R}^n : |y| < 1\}$, such that:

(i) For some $\delta > 0, \bigcup_{j=1}^{\infty} \Psi_j(\{y \in \mathbb{R}^n : |y| < 1/2\}) \supset \Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$, where $\Psi_j = \Phi_j^{-1}$.

(ii) For some $R \in \mathbb{N}$, every collection of $R+1$ of the sets U_j has empty intersection.

(iii) For each $j, \Phi_j(U_j \cap \Omega) = \{y \in B : y_n > 0\}$.

(iv) If $(\Phi_{j,1}, \ldots, \Phi_{j,n})$ and $(\Psi_{j,1}, \ldots, \Psi_{j,n})$ denote the components of Φ_j and Ψ_j, respectively, then there exists $M \in \mathbb{N}$ such that for all multiindices $\alpha : |\alpha| \leq m$, for every $i, 1 \leq i \leq n$, and for every j, we have
$|D^\alpha \Phi_{j,i}(x)| \leq M, \quad x \in U_j,$
$|D^\alpha \Psi_{j,i}(y)| \leq M, \quad y \in B.$"

Remark 1.0.19 The following assertion about interrelationship of these conditions holds true: (III) $\Rightarrow$ (II) $\Rightarrow$ (I).

The preceding definition described domains which are somehow regular, whereas the next one is an example of an irregular domain.

Definition 1.0.20 Consider domain $\Omega \subset \mathbb{R}^n$ with $(n-1)$-dimensional boundary such that Ω lies on only one side of its boundary. Ω is said to have an outer *cusp* at the point $x \in \partial\Omega$ if no finite open cone of positive volume contained in Ω can have vertex at x. These boundary cusps can be characterized by their sharpness. We say, that Ω has an *exponential cusp* at $x \in \partial\Omega$ if, as the term himself prompts, this cusp is of exponential sharpness. More precisely, if for every $k \in \mathbb{R}$, we have:

$$\lim_{r \to 0_+} \frac{\lambda_{n-1}(\partial B(x,r) \cap \Omega)}{r^k} = 0\,,$$

where λ_{n-1} means $(n-1)$-dimensional Lebesgue measure, i.e. surface area. Ω is said to have *standard cusp* at $x \in \partial\Omega$ if this cusp is of power sharpness, intended as specified therein before.

We won't list here the full range of existing embedding theorems for Sobolev spaces, we just summarize the core results.

Theorem 1.0.21 (A) *Let Ω be a domain in $\mathbb{R}^n$, Ω^d be the d-dimensional domain obtained by intersecting Ω with a d-dimensional plane in $\mathbb{R}^n$, $1 \leq d \leq n$ and $k \in \mathbb{N}_0$, $1 \leq p < \infty$.*

(I) *If Ω has the cone property, then there exist the following continuous embeddings:*

(i) *Suppose $kp < n$ and $n - kp < d \leq n$. Then:*
$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \quad p \leq q \leq \frac{np}{n-kp},$
$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega^d), \quad p \leq q \leq \frac{dp}{n-kp}.$

(ii) *Suppose $kp = n$ and $1 \leq d \leq n$. Then:*
$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega^d), \quad p \leq q < \infty.$

(iii) *Suppose $kp > n$. Then:*
$W^{k,p}(\Omega) \hookrightarrow C_B^0(\Omega)$.

(II) *If Ω has the strong local Lipschitz property, then, moreover, there exist the following continuous embeddings:*

(i) *Suppose $kp > n > (k-1)p$. Then:*
$W^{k,p}(\Omega) \hookrightarrow C^{0,\lambda}(\overline{\Omega}), \quad 0 < \lambda \leq k - \frac{n}{p}$.

(ii) *Suppose $n = (k-1)p$. Then:*
$W^{k,p}(\Omega) \hookrightarrow C^{0,\lambda}(\overline{\Omega}), \quad 0 < \lambda < 1$.

(III) *If Ω has the strong local Lipschitz property and is bounded and p satisfies $1 \leq p < n$, then there exist the following continuous embeddings:*
$W^{1,p}(\Omega) \hookrightarrow L^q(\partial\Omega), \quad 1 \leq q \leq \frac{np-p}{n-p}$.

(B) *Let Ω be a domain in $\mathbb{R}^n$, Ω_0 a bounded subdomain of Ω and Ω_0^d the intersection of Ω_0 with a d-dimensional plane in $\mathbb{R}^n$. Let $k \in \mathbb{N}, k \leq 1$ and $1 \leq p < \infty$.*

(I) *If Ω has the cone property, then there exist the following compact embeddings:*

(i) *Suppose $kp \leq n$ and $0 < n - kp < d \leq n$. Then:*
$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega_0^d), \quad 1 \leq q < \frac{dp}{n-kp}$.

(ii) *Suppose $kp = n$ and $1 \leq d \leq n$. Then:*
$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega_0^d), \quad 1 \leq q < \infty$.

(iii) *Suppose $kp > n$. Then:*
$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega_0^d), \quad 1 \leq q \leq \frac{dp}{n-kp}$,
$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow C_B^0(\Omega)$.

(II) *If Ω has the strong local Lipschitz property, then, moreover, there exist the following compact embeddings:*

(i) *Suppose $kp > n > (k-1)p$. Then:*
$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow C^{0,\lambda}(\overline{\Omega_0}), \quad 0 < \lambda < k - \frac{n}{p}$.

(ii) *Suppose $kp > n$. Then:*
$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow C^0(\overline{\Omega_0})$.

The embeddings presented in (A), except trace embeddings, can be in some way extended to the domains with boundary irregularities comparable to standard cusps. But as soon as we consider more irregular domains, e.g. with sufficiently sharp boundary cusp, the theorem fails. For example, for a domain with an exponential cusp we obtain that $W^{k,p}(\Omega)$ is not embedded in $L^q(\Omega)$ for any $q > p$.

Now, we will focus on the elementary embedding $W^{k,p}(\Omega) \hookrightarrow L^p(\Omega)$, existence of which is clear on an arbitrary domain Ω for any $k \in \mathbb{N}_0$ and $1 \leq p \leq \infty$. For values $k = 1$ and $p = 2$, the nature of this embedding determines spectral properties of the Neumann Laplacian operator in $L^2(\Omega)$, describing one of the problems studied in PDEs. An important quantity in this context is

$$\beta(I) := \inf\{\|I - P\| : P \in \mathcal{F}(W^{1,2}(\Omega), L^2(\Omega))\},$$

where I is the embedding map of $W^{1,2}(\Omega)$ into $L^2(\Omega)$ and $\mathcal{F}(W^{1,2}(\Omega), L^2(\Omega))$ denotes the set of linear operators from $W^{1,2}(\Omega)$ into $L^2(\Omega)$ which are bounded and have the finite rank. It is obvious that $0 \leq \beta(I) \leq 1$. There exist various sufficient conditions guaranteeing $\beta(I)$ to take or not to take the boundary values, i.e. 0 or 1. Amick showed that $0 \leq \beta(I) < 1$ if and only if the 2,2-*Poincaré inequality* is satisfied on Ω, that is, there exists a positive constant C depending on Ω such that

$$\int_\Omega |f(x) - f_\Omega|^2 dx \leq C \int_\Omega |\nabla f(x)|^2 dx \text{ for every } f \in W^{1,2}(\Omega),$$

where f_Ω denotes the integral average[4] of f over Ω. Motivated by Amick's observation, W.D. Evans and D.J. Harris have searched for an effective criterion for determination, whether the Poincaré inequality is satisfied on rather irregular domains. They brought out the theorem asserting that for a special class of irregular domains, this problem on a domain is equivalent with an analogous but more accessible one on an interval in $\mathbb{R}$ provided with measure which, in some sense, measures the irregularity of the boundary of the domain. Of course, to this end, some kind of relevant correspondence between the domain and the interval under consideration is necessary. Therefore that class of irregular domains from theorem is chosen to consist of domains possessing what is called a *generalized ridge*, this being a Lipschitz curve which, roughly speaking, forms a central axis of the domain. For a desired interval, the preimage of the ridge of a domain is taken then. To this class belong, for instance, trumpet-shaped domains, horn-shaped domains or "rooms and passages". Let us recall the results of Evans and Harris presented in [2]:

"**Notation.** We denote by B(x,r) the open ball $\{y : |y - x| < r\}$ in $\mathbb{R}^n$, where $|\cdot|$ is any norm on $\mathbb{R}^n$; once this norm is chosen, it must remain fixed thereafter. Also we denote by r_n the best constant such that $|a \cdot b| \leq r_n |a||b|$ for all $a, b \in \mathbb{R}^n$, $a \cdot b$ being the scalar product $\sum_{j=1}^n a_j b_j$ for $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$. Let us yet recall that a function g which is Lipschitz on an interval J is differentiable almost everywhere, and for all $t \in J$,

$$G(t) := \limsup_{n \to \infty} (n(g(t + 1/n) - g(t)))$$

exists, $G(t)$ being equal to $g'(t)$ whenever it exists. We shall define $g'(t)$ for all $t \in J$ by setting $g'(t) = G(t)$. The set of functions $F \in \mathrm{Lip}_{\mathrm{loc}}(J)$ such that $F, F' \in L^p(J, d\mu)$ will be denoted by $L^{1,p}(J, d\mu)$.

Definition 1.0.22 A domain Ω will be called a *generalized ridge domain* if there exist functions u, ψ, τ, an interval $J = [a, b)$, where $b \leq \infty$, and positive constants $\alpha, \beta, \gamma, \delta$ such that the following conditions are satisfied:

(i) $u : J \to \Omega$, $\psi : J \to \mathbb{R}^+ = (0, \infty)$ are Lipschitz;

(ii) $\tau : \Omega \to J$ and for each $x \in \Omega$ there exists a neighborhood $V(x) \subset \Omega$ such that for all $y, z \in V(x), |\tau(z) - \tau(y)| \leq \gamma |z - y|$; that is, τ is uniformly locally Lipschitz on Ω;

(iii) $|x - u \circ \tau(x)| \leq \alpha\, \psi \circ \tau(x)$ for all $x \in \Omega$;

(iv) $r_n(|u'(t)| + |\psi'(t)|) \leq \beta$ for all $t \in J$;

[4]Throughout the thesis, we shall use the notation $f_\Omega = \fint_\Omega f(x) d\mu(x) = \frac{1}{\mu(\Omega)} \int_\Omega f(x) d\mu(x)$ for Ω being an arbitrary subset of a space endowed with measure μ.

(v) with $B_t := B(u(t), \psi(t))$ and $\mathfrak{C}(x) := \{y : sy + (1-s)x \in \Omega \text{ for all } 0 \le s \le 1\}$ we have that for all $x \in \Omega$, $B_{\tau(x)} \subset \Omega$ and $\mathfrak{C}(x) \cap B_{\tau(x)}$ contains a ball $B(x)$ such that $\lambda(B(x))/\lambda(B_{\tau(x)}) \ge \delta > 0$.

The curve $t \to u(t) : J \to \Omega$ will be called a *generalized ridge* of Ω."

The following theorem is the main result of studies in [2].

"**Theorem 1.0.23** *Let $1 < p < \infty$ and suppose that $\Omega_1 = \tau^{-1}(J_1)$, where J_1 is a measurable subset of J, satisfies*

$$r_n \sup_{\Omega_1}\{\psi \circ \tau(x)\} =: k(\Omega_1) < \infty.$$

Then the following two statements are equivalent:

(a) *for some constant $C(\Omega_1) > 0$,*

$$\|f - f_{\Omega_1}\|_{L^p(\Omega_1)} \le C(\Omega_1)\|\nabla f\|_{L^p(\Omega)} \qquad (f \in W^{1,p}(\Omega));$$

(b) *for some constant $c(J_1) > 0$ and $F_{J_1} := \mu(J_1)^{-1} \int_{J_1} F(t)d\mu(t)$,*

$$\|F - F_{J_1}\|_{L^p(J_1,d\mu)} \le c(J_1)\|F'\|_{L^p(J,d\mu)} \qquad (F \in L^{1,p}(J, d\mu)).$$

The least constants $C(\Omega_1)$, $c(J_1)$ satisfy

$$\gamma^{-1}c(J_1) \le C(\Omega_1) \le 2\delta^{-1}k(\Omega_1)(\alpha+1)^n c_p\{\alpha + 1 + 2^{n+1}c_p\} + 2c(J_1)\beta(\alpha+1)^n c_p."$$

As formerly noted, there exist plenty of generalizations and extensions of the classical Sobolev spaces $W^{k,p}(\Omega)$, involving spaces $W^{s,p}(\Omega)$ for an arbitrary real value of s, weighted spaces that introduce weight functions into the L^p-norms, spaces which allow different orders of differentiation and different L^p-norms in the various coordinate directions (anisotropic spaces), Orlicz-Sobolev spaces modeled on the generalizations of L^p-spaces known as "Orlicz spaces" and Sobolev spaces built upon metric spaces equipped with a measure. The remaining chapters of this thesis will be devoted to the last one mentioned.

Chapter 2

Sobolev spaces on metric measure spaces

2.1 Metric measure spaces

In what follows, by a metric measure space $(X, d, \boldsymbol{m})$ we mean a metric space (X, d) equipped with a nonnegative outer Borel-regular measure $\boldsymbol{m}$, i.e. such measure that all sets $Y \subset X$ are $\boldsymbol{m}$-measurable and for each $Y \subset X$ there exists a Borel set (set belonging to the σ-algebra generated by all open subsets of X) Z such that $Y \subset Z$ and $\boldsymbol{m}(Y) = \boldsymbol{m}(Z)$. In addition to this, we always assume, that $0 < \boldsymbol{m}(B) < \infty$ for every ball $B \subset X$ and that balls are open. Our last condition implies $\sigma-$finiteness of $\boldsymbol{m}$, that is that $X = \bigcup_{j=1}^{\infty} U_j$, where U_j are open subsets of X having finite measure.

Notation. Assume $(X, d, \boldsymbol{m})$ is a metric measure space. Let us just recall the formerly settled notation of the integral average of function f over an arbitrary set $\Omega \subset X$: $f_\Omega = \fint_\Omega f(x) d\boldsymbol{m}(x) = \frac{1}{\boldsymbol{m}(\Omega)} \int_\Omega f(x) d\boldsymbol{m}(x)$. Next we adopt the convention that for any ball $B \subset X$ and $\sigma > 0$, σB denotes ball in X with the same center as B and with radius equal σ times radius of B.

Definition 2.1.1 Consider metric measure space $(X, d, \boldsymbol{m})$. Measure $\boldsymbol{m}$ is *doubling* if there is a constant $C_{\boldsymbol{m}} \geq 1$ (called *doubling constant*) such that

$$\boldsymbol{m}(2B) \leq C_{\boldsymbol{m}} \boldsymbol{m}(B),$$

whenever B is a ball in X. Metric spaces equipped with a doubling measure are called *spaces of homogeneous type* (we abbreviate "homogeneous spaces") and the quantity $s = \log_2 C_{\boldsymbol{m}}$ is called *homogeneous dimension.*

Doubling measure $\boldsymbol{m}$ obviously satisfies $\boldsymbol{m}(\sigma B) \leq C(\boldsymbol{m}, \sigma)\boldsymbol{m}(B)$ for any ball $B \subset X$ and $\sigma \geq 1$, where $C(\boldsymbol{m}, \sigma)$ is a constant depending on $\boldsymbol{m}$ (actually on $C_{\boldsymbol{m}}$) and σ only. Homogeneous dimension is not uniquely associated with given doubling measure as we can always take $C_{\boldsymbol{m}}$ larger. In the case of space $\mathbb{R}^n$ equipped with the Lebesgue measure, we have $C_{\boldsymbol{m}} = 2^n$ and hence $s = n$. Spaces of homogeneous type abound in quite rich theory, involving sufficient and necessary conditions for space to be of homogeneous type, properties similar to those of Lebesgue measure on $\mathbb{R}^n$ (estimates for Hardy-Littlewood maximal function, Lebesgue differentiation theorem)[1] or

[1] Both to be presented later.

interesting subclasses of doubling measures (so called s-regular measures). We state here just one feature of doubling measures, which proves to be useful in our further work.

Lemma 2.1.2 *If $(X, d, \boldsymbol{m})$ is a metric measure space with doubling measure, $C_{\boldsymbol{m}}$ is a doubling constant and $s = \log_2 C_{\boldsymbol{m}}$, then*

$$\frac{\boldsymbol{m}(B(x,r))}{\boldsymbol{m}(B_0)} \geq 4^{-s}\left(\frac{r}{r_0}\right)^s,$$

whenever $B_0 \subset X$ is a ball of radius r_0, $x \in B_0$ and $r \leq r_0$.

We have already mentioned *Hardy-Littlewood maximal function* in what precedes. Here we present closer insight into this notion because it turns out to be indispensable in the closing sections of this chapter.

Definition 2.1.3 Let $(X, d, \boldsymbol{m})$ be a homogeneous space. For a locally integrable real-valued function f on X (written $f \in L^1_{\mathrm{loc}}(X)$) we define *maximal function* on X

$$\mathcal{M}(f)(x) = \sup_{r>0} ⨍_{B(x,r)} |f| d\boldsymbol{m},$$

and *restricted maximal function* on X

$$\mathcal{M}_R(f)(x) = \sup_{0<r<R} ⨍_{B(x,r)} |f| d\boldsymbol{m}.$$

Observe that $\mathcal{M}$ is subadditive operator, that means that $\mathcal{M}(f+g)(x) \leq \mathcal{M}(f)(x) + \mathcal{M}(g)(x)$ for every $f, g \in L^1_{\mathrm{loc}}(X)$ and $x \in X$.
Obviously, $\mathcal{M}(f) \leq \|f\|_{L^\infty(X)}$. On the other hand, Lebesgue's differentiation theorem, which asserts that for any $f \in L^1_{\mathrm{loc}}(X)$ equality

$$\lim_{r \to 0} ⨍_{B(x,r)} |f| d\boldsymbol{m} = |f(x)|$$

holds true for almost every $x \in X$, leads to a lower bound of $\mathcal{M}(f)$. Indeed, the maximal function $\mathcal{M}(f)$ is always at least as large as $|f|$ $\boldsymbol{m}-$a.e. on X.

Theorem 2.1.4 *Operator $\mathcal{M}$ maps*

(i) *$L^1(X)$ to weak-$L^1(X)$, precisely*[2]

$$\boldsymbol{m}(\{x \in X : \mathcal{M}(f)(x) > t\}) \leq \frac{C}{t} \int_X |f| d\boldsymbol{m}$$

for all $t > 0$ and $f \in L^1(X)$. Constant C depends only on the doubling constant $C_{\boldsymbol{m}}$.

(ii) *$L^p(X)$ to $L^p(X)$ and*

$$\int_X |\mathcal{M}(f)|^p d\boldsymbol{m} \leq C \int_X |f|^p d\boldsymbol{m}$$

whenever $f \in L^p(X)$. Constant C depends on p and the doubling constant $C_{\boldsymbol{m}}$ only.

[2] Function f belongs to *weak-$L^p(X)$* space, where $p > 0$, if there is a constant $c > 0$ such that $\boldsymbol{m}(\{x \in X : |f(x)| > t\}) \leq ct^{-p}$ for all $t > 0$.

Let's denote by $\mathcal{B}(x)$ the set of all balls in X containing x and define

$$\widetilde{\mathcal{M}}(f)(x) = \sup_{B \in \mathcal{B}(x)} \fint_B |f| d\boldsymbol{m}$$

for any $f \in L^1_{\text{loc}}(X)$ and $x \in X$. Then $\widetilde{\mathcal{M}}(f) \approx \mathcal{M}(f)$, so $\widetilde{\mathcal{M}}(f)$ and $\widetilde{\mathcal{M}}$ have similar features to those of $\mathcal{M}(f)$ and $\mathcal{M}$ respectively.

2.2 Sobolev spaces $N^{1,p}$

The first version of the Sobolev-type spaces on metric measure spaces is inspired by the characterization of classical Sobolev spaces in terms of absolute continuity on lines. We shall use this approach to formulate our main theorem.

Definition 2.2.1 A function f is *absolutely continuous* on an interval $[a, b] \subset \mathbb{R}$, if $f(x) = c + \int_a^x h(t)dt$ for some $c \in \mathbb{R}$, $h \in L^1([a, b])$ and all $x \in [a, b]$. A function f is *locally absolutely continuous* on an open set $U \subset \mathbb{R}$ if it is absolutely continuous on each closed interval $[a, b] \subset U$. Finally, we say that f is *absolutely continuous on lines* on an open set $\Omega \subset \mathbb{R}^n$, written $f \in ACL(\Omega)$, if f is Borel[3] and for almost every line l parallel to one of the coordinate axes, the restriction of f to $l \cap \Omega$ is locally absolutely continuous on $l \cap \Omega$.

An absolutely continuous function f is differentiable λ-a.e. on interval (a, b) (moreover $f' = h$ λ-a.e. on (a, b)), therefore $f \in ACL(\Omega)$ has partial derivatives λ_n-a.e. on Ω. Consequently, ∇f is defined λ_n-a.e. on Ω.

Theorem 2.2.2 *If $\Omega \subset \mathbb{R}^n$ is an open set and $1 \le p < \infty$, then*

$$W^{1,p}(\Omega) = \{f \in L^p(\Omega) \cap ACL(\Omega) : \nabla f \in L^p(\Omega)\}.$$

This should be interpreted as follows: each element of $W^{1,p}(\Omega)$ (an equivalence class) has representative belonging to set on the right hand side of the equation and, conversely, for every function from this set, its classical partial derivatives are equal to weak partial derivatives.

Since the notion of almost every line parallel to one of the coordinate axes and the notion of the gradient are strictly Euclidean, we have to find their generalizations to all metric measure spaces. To this end, one first needs to develop the theory of curves in metric spaces.

Definition 2.2.3 Let (X, d) be a metric space. A *curve* in X is any continuous mapping $\gamma : [a, b] \to X$. For image of the curve we adopt the notation $\langle\gamma\rangle = \gamma([a, b])$. By *length of the curve* γ we understand the quantity

$$l(\gamma) := \sup\{\sum_{i=0}^{m-1} d(\gamma(t_i), \gamma(t_{i-1}))\},$$

where the supremum is taken over all partitions of interval $[a, b]$, i.e. finite sequences $a = t_0 < t_1 < \ldots < t_m = b$. The curve is *rectifiable* if its length is finite.

[3]Function is said to be Borel on a measure space (X, S, μ), where S contains all Borel sets, if the preimage of every open set in $\mathbb{R}$ is a Borel set in X.

The following theorem actually claims, that every rectifiable curve γ admits a 1 - Lipschitz parametrization $\widetilde{\gamma}$.

Theorem 2.2.4 *If $\gamma : [a,b] \to X$ is a rectifiable curve, then there exists a unique curve $\widetilde{\gamma} : [0, l(\gamma)] \to X$ such that $\gamma = \widetilde{\gamma} \circ s_\gamma$, where $s_\gamma : [a,b] \to [0, l(\gamma)]$ is given by $s_\gamma(t) = l(\gamma|_{[a,t]})$. Furthermore, $\widetilde{\gamma}$ is rectifiable and $l(\widetilde{\gamma}|_{[0,t]}) = t$ for each $t \in [0, l(\gamma)]$. In particular, $\widetilde{\gamma}$ is a 1 - Lipschitz mapping.*

Remark 2.2.5 We call $\langle \widetilde{\gamma} \rangle$ parametrized by the *arc-length* because $l(\widetilde{\gamma}|_{[0,t]}) = t$ for each $t \in [0, l(\gamma)]$.

We use the existence of arc-length parametrization of rectifiable curves to define the integral of a Borel function along a rectifiable curve.

Definition 2.2.6 Let $\gamma : [a,b] \to X$ be a rectifiable curve, $\widetilde{\gamma} : [0, l(\gamma)] \to X$ be its arc-length parametrization and let $f : \langle \gamma \rangle \to [0, \infty]$ be a Borel function. Then we define

$$\int_{\langle \gamma \rangle} f := \int_0^{l(\gamma)} f(\widetilde{\gamma}(t)) dt.$$

Denote by $\mathfrak{M}$ the family of all nonconstant rectifiable curves in a metric measure space $(X, d, \boldsymbol{m})$. We wish to set an outer measure on $\mathfrak{M}$ in order to be able to speak about subfamilies of $\mathfrak{M}$ with measure zero. Withal, we pay attention to keep the notion of 'a.e. curve with respect to new created measure on $\mathfrak{M}$' consistent with the notion of 'almost every line parallel to a given coordinate direction' in the Euclidean spaces.

Definition 2.2.7 For $\Gamma \subset \mathfrak{M}$, let $F(\Gamma)$ be the family of all Borel functions $f : X \to [0, \infty]$ such that

$$\int_{\langle \gamma \rangle} f \geq 1 \text{ for every } \gamma \in \Gamma.$$

For $1 \leq p < \infty$ we define

$$\mathrm{Mod}_p(\Gamma) = \inf_{f \in F(\Gamma)} \int_X f^p d\boldsymbol{m}.$$

The number $\mathrm{Mod}_p(\Gamma)$ is called the *p-modulus* of the family Γ. If some property holds for all curves $\gamma \in \mathfrak{M} \backslash \Gamma$, where $\mathrm{Mod}_p(\Gamma) = 0$, then we say that property holds for *p-a.e. curve.*

It can be proved that Mod_p meets our requirements, i.e. it is an outer measure on $\mathfrak{M}$ and, if not strictly speaking, the notion of 'p-a.e. curve' appropriately generalizes the notion of 'almost every line' in $\mathbb{R}^n$.

Theorem 2.2.8 *Let $\Gamma \subset \mathfrak{M}$. Then $Mod_p(\Gamma) = 0$ if and only if there exists a Borel function $0 \leq f \in L^p(X)$ such that*

$$\int_{\langle \gamma \rangle} f = +\infty \textit{ for every } \gamma \in \Gamma.$$

In particular, for any $1 \leq p < \infty$ and Borel function $0 \leq f \in L^p(X)$,

$$\int_{\langle \gamma \rangle} f < \infty \textit{ for } p\textit{-a.e. } \gamma \in \mathfrak{M}.$$

We are now ready to introduce an upper gradient, the crucial ingredient of the theory of generalized Sobolev spaces.

Definition 2.2.9 Let $f : X \to \mathbb{R}$ be a Borel function. A Borel function $g : X \to [0, \infty]$ is an *upper gradient* of f if

$$|f(\gamma(a)) - f(\gamma(b))| \leq \int_{\langle\gamma\rangle} g \tag{2.1}$$

for every rectifiable curve $\gamma : [a, b] \to X$. We say that g is a *p-weak upper gradient* of f if (2.1) holds for p-a.e. curve $\gamma \in \mathfrak{M}$.

We will recall here some important results for upper gradients and p-weak upper gradients, among others showing that the notion of the upper gradient is a natural generalization of the norm of the classical gradient.

Lemma 2.2.10 *If $f : X \to \mathbb{R}$ is a Borel function, g is its p-weak upper gradient and $\widetilde{g}$ is a Borel function such that $\widetilde{g} = g$ $\boldsymbol{m}$-a.e. on X, then $\widetilde{g}$ is a p-weak upper gradient of f too.*

Remark 2.2.11 If we suppose that g in Lemma 2.2.10 is even an upper gradient of f, $\widetilde{g}$ may no longer be an upper gradient of f and remains still just a p-weak upper gradient of f.

Lemma 2.2.12 *If $f : X \to \mathbb{R}$ is a Borel function, g is its p-weak upper gradient which is finite $\boldsymbol{m}$-a.e. on X, then for every $\varepsilon > 0$ there is an upper gradient g_ε of f such that*

$$g_\varepsilon \geq g \quad \text{everywhere, and} \quad \|g_\varepsilon - g\|_{L^p(X)} < \varepsilon.$$

Theorem 2.2.13 *If $f \in C^\infty(\Omega)$, where Ω is an open set in $\mathbb{R}^n$, then $|\nabla f|$ is an upper gradient of f. Moreover, $|\nabla f|$ is the least one among upper gradients belonging to $L^1_{\mathrm{loc}}(\Omega)$ in the sense that if $g \in L^1_{\mathrm{loc}}(\Omega)$ is another upper gradient of f, then $g \geq |\nabla f|$ $\boldsymbol{m}$-a.e. on Ω.*

It's time to come up to Sobolev-type spaces $N^{1,p}$ themselves.

Definition 2.2.14 Let $(X, d, \boldsymbol{m})$ be a metric measure space, $1 \leq p < \infty$. Denote by $\widetilde{N}^{1,p}(X, d, \boldsymbol{m})$ the class of all Borel functions[4] on X belonging to $L^p(X)$ for which there exists a p-weak upper gradient in $L^p(X)$. Next, we define a functional on $\widetilde{N}^{1,p}(X, d, \boldsymbol{m})$ as follows:

$$\|f\|_{\widetilde{N}^{1,p}} = \|f\|_{L^p} + \inf_g \|g\|_{L^p} \quad \forall f \in \widetilde{N}^{1,p}(X, d, \boldsymbol{m}),$$

where the infimum is taken over all p-weak upper gradients g of f. We establish an equivalence relation $\sim$ in $\widetilde{N}^{1,p}(X, d, \boldsymbol{m})$, f_1, f_2 being equivalent if $\|f_1 - f_2\|_{\widetilde{N}^{1,p}} = 0$. Finally, we define the normed linear space $N^{1,p}(X, d, \boldsymbol{m})$ as the quotient $\widetilde{N}^{1,p}(X, d, \boldsymbol{m})/\sim$ equipped with the norm $\|[f]\|_{N^{1,p}} := \|f\|_{\widetilde{N}^{1,p}}$ for $f \in \widetilde{N}^{1,p}(X, d, \boldsymbol{m})$, i.e. equivalence class $[f] \in N^{1,p}(X, d, \boldsymbol{m})$.

[4] We would like to emphasize here that functions in $\widetilde{N}^{1,p}(X, d, \boldsymbol{m})$ are defined everywhere and not only up to sets of measure zero.

Theorem 2.2.15 *If $f_1, f_2 \in \widetilde{N}^{1,p}(X, d, \boldsymbol{m})$, $f_1 = f_2$ $\boldsymbol{m}-$a.e. on X, then $f_1 \sim f_2$.*

Theorem 2.2.16 *For $1 \leq p < \infty, N^{1,p}(X, d, \boldsymbol{m})$ is a Banach space.*

Theorem 2.2.17 *If $\Omega \subset \mathbb{R}^n$ is an open set and $1 \leq p < \infty$, then*

$$N^{1,p}(\Omega, |\cdot|, \lambda_n) = W^{1,p}(\Omega).$$

The equation should be interpreted as the equality of two Banach spaces, i.e. sets are equal and norms are equivalent.

Remarks 2.2.18 Here $|\cdot|$ is the Euclidean metric on $\mathbb{R}^n$ and λ_n is the Lebesgue measure on $\mathbb{R}^n$. However, we have defined the $N^{1,p}$ spaces only over metric measure spaces specified at the very beginning of this chapter. So to be rigorous, we can replace the Lebesgue measure by the Lebesgue outer measure and all assumptions of "our" metric measure spaces will be satisfied.
Let us give further explanation of the theorem in order to be correctly interpreted. In fact, each function f from $\widetilde{N}^{1,p}(\Omega, |\cdot|, \lambda_n)$ belongs also to $ACL(\Omega)$ and, moreover, $|\nabla f| \leq g$ λ_n-a.e. on Ω for any locally integrable (consequently for any L^p integrable) p-weak upper gradient g of f. This inequality leads immediately to several results. First of all, $|\nabla f| \in L^p(\Omega)$, so $f \in W^{1,p}(\Omega)$. (This is the spot where we use characterization of Sobolev spaces via absolute continuity on lines.) In addition to this, we obtain $\|f\|_{W^{1,p}} \leq \|f\|_{\widetilde{N}^{1,p}}$. This also shows that if $f_1 \sim f_2$ in $\widetilde{N}^{1,p}(\Omega, |\cdot|, \lambda_n)$, then both f_1 and f_2 determine the same element of $W^{1,p}(\Omega)$. Due to Theorem 2.2.15, functions from different equivalence classes as elements of $N^{1,p}(\Omega, |\cdot|, \lambda_n)$ never represent the same equivalence class in $W^{1,p}(\Omega)$. Thus the inclusion $N^{1,p}(\Omega, |\cdot|, \lambda_n) \subset W^{1,p}(\Omega)$ is clear. The remaining inclusion should be understood as that for every element of $W^{1,p}(\Omega)$ there exists a representative f for which $|\nabla f|$ (here, weak partial derivatives are considered) is a p-weak upper gradient and $\|f\|_{N^{1,p}} \leq \|f\|_{W^{1,p}}$.

The proof of the preceding theorem implies also the following claim.

Corollary 2.2.19 *Any $f \in W^{1,p}(\Omega)$, $1 \leq p < \infty$ has a representative for which $|\nabla f|$ is a p-weak upper gradient. On the other hand, if $g \in L^1_{\text{loc}}(\Omega)$ is a p-weak upper gradient of f, then $g \geq |\nabla f|$ λ_n-a.e. on Ω.*

Theorem 2.2.20 *For every $f \in N^{1,p}(X, d, \boldsymbol{m})$, $1 \leq p < \infty$, there exists the least p-weak upper gradient $g_f \in L^p(X)$ of f. That means that if $g \in L^p(X)$ is another p-weak upper gradient of f, then $g \geq g_f$ $\boldsymbol{m}-$a.e. on X.*

A disadvantage of this definition of a Sobolev-type space over a metric measure space is that the theory becomes trivial for metric measure spaces whose structure isn't rich enough. For instance, if space X doesn't contain any nonconstant rectifiable curves, then $N^{1,p}(X) = L^p(X)$. This unpleasant situation is a consequence of the fact that on spaces with this feature, $g \equiv 0$ is an upper gradient of every Borel function (and, in general on spaces with $\sigma-$finite Borel measure, each L^p function can be changed on a set of measure zero in a way that the resulting function is Borel). This inconvenience of $N^{1,p}$ spaces is not insignificant since the class of concerned metric measure spaces involves also such important examples as fractals, e.g. Van Koch snowflake or Cantor type sets. However, there exist other approaches to Sobolev spaces in setting of metric measure spaces that do not fail in this respect.

2.3 Sobolev spaces $M^{1,p}$

The following characterization of classical Sobolev spaces doesn't involve derivatives, therefore it can be conveniently used in a more general setting of Sobolev spaces, whose theory is rich enough even in case of metric measure spaces containing constant rectifiable curves only.

Theorem 2.3.1 *Let $\Omega = \mathbb{R}^n$ or $\Omega \subset \mathbb{R}^n$ be a bounded domain with strong local Lipschitz property* (see Definition 1.0.18) *and $1 < p < \infty$. Then $f \in W^{1,p}(\Omega)$ if and only if $f \in L^p(\Omega)$ and there exists $0 \leq g \in L^p(\Omega)$ so that*

$$|f(x) - f(y)| \leq |x - y|(g(x) + g(y)) \quad \lambda_n - a.e. \text{ on } \Omega. \tag{2.2}$$

Moreover $\|\nabla f\|_{L^p} \approx \inf_g \|g\|_{L^p}$, where the infimum is taken over the class of all functions g satisfying (2.2).

Throughout this section, $(X, d, \boldsymbol{m})$ is a metric measure space.

Definition 2.3.2 For $0 < p < \infty$ we define $M^{1,p}(X, d, \boldsymbol{m})$ to be the set of all functions $f \in L^p(X)$ for which there exists a function $0 \leq g \in L^p(X)$ such that

$$|f(x) - f(y)| \leq d(x, y)(g(x) + g(y)) \quad \boldsymbol{m} - \text{a.e. on } X. \tag{2.3}$$

Note that g can be considered to be a Borel function. Thus, if we denote by $D(f)$ the class of all nonnegative Borel functions g satisfying (2.3), we have that
$f \in M^{1,p}(X, d, \boldsymbol{m})$ if and only if $f \in L^p$ and $D(f) \cap L^p(X) \neq \emptyset$.
For each $f \in M^{1,p}(X, d, \boldsymbol{m})$ define

$$\|f\|_{M^{1,p}} = \|f\|_{L^p} + \inf_{g \in D(f)} \|g\|_{L^p}.$$

Theorem 2.3.3 *$M^{1,p}(X, d, \boldsymbol{m})$, where $0 < p < \infty$, is a linear space. Furthermore, if $1 \leq p < \infty$, the functional $\|\cdot\|_{M^{1,p}}$ is a norm on $M^{1,p}(X, d, \boldsymbol{m})$ and $M^{1,p}(X, d, \boldsymbol{m})$ equipped with this norm is a Banach space.*

As an direct consequence of Theorem 2.3.1 we obtain an assertion about coincidence of $M^{1,p}$ spaces with $W^{1,p}$ spaces on certain domains in $\mathbb{R}^n$.

Theorem 2.3.4 *Suppose that $1 < p < \infty$ and that $\Omega = \mathbb{R}^n$ or $\Omega \subset \mathbb{R}^n$ is a bounded domain with strong local Lipschitz property. Then $W^{1,p}(\Omega) = M^{1,p}(\Omega, |\cdot|, \lambda_n)$. The equation is intended as an equality of two Banach spaces, i.e. sets are equal and norms are equivalent.*

For further information about the space $(\Omega, |\cdot|, \lambda_n)$ see Remarks 2.2.18.

Theorem 2.3.5 *Let $0 < p < \infty$. Then for every $f \in M^{1,p}(X, d, \boldsymbol{m})$ and $\varepsilon > 0$ there is a Lipschitz function φ on X such that*

$$\boldsymbol{m}(\{x \in X : f(x) \neq \varphi(x)\}) < \varepsilon,$$

and

$$\|f - \varphi\|_{M^{1,p}} < \varepsilon.$$

The above outcome can be regarded as some counterpart of results on density of smooth functions in the standard Sobolev spaces. One of the merits of the theory of $M^{1,p}$ spaces is that the very important theorem dealing with the embedding properties of classical Sobolev spaces (see Theorem 1.0.21) has extension to the setting of $M^{1,p}$ spaces. The nature of the embeddings of space $W^{k,p}(\Omega)$ depends on domain Ω and on relation between p and the dimension of the Euclidean space. As an analogue of the dimension in general metric measure space will serve us the lower bound for the growth of the measure.

Definition 2.3.6 We say that the measure $\boldsymbol{m}$ satisfies the condition $V(\sigma B_0, s, b)$ if

$$\boldsymbol{m}(B(x,r)) \geq br^s \quad \text{for every} \quad B(x,r) \subset \sigma B_0,$$

where $s, b > 0, \sigma \geq 1$ are fixed constants and $B_0 \subset X$ is a fixed ball.

Theorem 2.3.7 *Let $(X, d, \boldsymbol{m})$ be a metric measure space, $B_0 \subset X$ be a fixed ball of radius r_0, $0 < p < \infty$, $\sigma > 1$. Assume that the measure $\boldsymbol{m}$ satisfies the condition $V(\sigma B_0, s, b)$. Then there exist constants C, C_1 and C_2 depending only on s, p and σ such that for every $f \in M^{1,p}(X, d, \boldsymbol{m})$ and $g \in D(f)$ we have*

(i) *If $0 < p < s$ and $p* = \frac{sp}{s-p}$, then $f \in L^{p*}(B_0)$ and*

$$\inf_{c \in \mathbb{R}} \Big(\fint_{B_0} |f - c|^{p*} d\boldsymbol{m} \Big)^{1/p*} \leq C \Big(\frac{\boldsymbol{m}(\sigma B_0)}{br_o^s} \Big)^{1/p} r_o \Big(\fint_{\sigma B_0} g^p d\boldsymbol{m} \Big)^{1/p}.$$

(ii) *If $p = s$, then*

$$\fint_{B_0} \exp\Big(C_1 b^{\frac{1}{s}} \frac{|f - f_{B_0}|}{\|g\|_{L^s(\sigma B_0)}} \Big) d\boldsymbol{m} \leq C_2.$$

(iii) *If $p > s$, then*

$$\|f - f_{B_0}\|_{L^\infty(B_0)} \leq C \Big(\frac{\boldsymbol{m}(\sigma B_0)}{br_o^s} \Big)^{1/p} r_o \Big(\fint_{\sigma B_0} g^p d\boldsymbol{m} \Big)^{1/p}.$$

In particular, there exists $\widetilde{f} = f\, \boldsymbol{m}-a.e.$ on σB_0 such that

$$|\widetilde{f}(x) - \widetilde{f}(y)| \leq C b^{-\frac{1}{p}} d(x,y)^{1-\frac{s}{p}} \Big(\int_{\sigma B_0} g^p d\boldsymbol{m} \Big)^{1/p} \quad \textit{for } x, y \in B_0,$$

i.e. $\widetilde{f}$ is Hölder continuous on B_0.

Remark 2.3.8 If $p* \geq 1$, then inequality in (i) can be replaced by

$$\Big(\fint_{B_0} |f - f_{B_0}|^{p*} d\boldsymbol{m} \Big)^{1/p*} \leq C \Big(\frac{\boldsymbol{m}(\sigma B_0)}{br_o^s} \Big)^{1/p} r_o \Big(\fint_{\sigma B_0} g^p d\boldsymbol{m} \Big)^{1/p},$$

since, in general, for $p \geq 1$ and $\Gamma \subset X$ with $\boldsymbol{m}(\Gamma) < \infty$ we have

$$\inf_{c \in \mathbb{R}} \Big(\fint_{\Gamma} |f - c|^p d\boldsymbol{m} \Big)^{1/p} \leq \Big(\fint_{\Gamma} |f - f_\Gamma|^p d\boldsymbol{m} \Big)^{1/p} \leq 2 \inf_{c \in \mathbb{R}} \Big(\fint_{\Gamma} |f - c|^p d\boldsymbol{m} \Big)^{1/p}.$$

Condition $p* \geq 1$ is necessary, since otherwise f need not be integrable on B_0.

Lemma 2.1.2 implies that every doubling measure $\boldsymbol{m}$ satisfies the $V(\sigma B_0, s, b)$ condition for each ball $B_0 \subset X$ and each $\sigma \geq 1$ with $s = \log_s C_{\boldsymbol{m}}$ and $b = 4^{-s}\sigma^{-s}r_o^{-s}\boldsymbol{m}(\sigma B_0)$. This leads to the following corollary.

Corollary 2.3.9 *Assume that $(X, d, \boldsymbol{m})$ is a homogeneous space with doubling constant $C_{\boldsymbol{m}}$ and with the associated homogeneous dimension $s = \log_s C_{\boldsymbol{m}}$. Let $B_0 \subset X$ be a fixed ball of radius r_o, $0 < p < \infty$, $\sigma > 1$. Then there exist constants C, C_1 and C_2 depending on p, $C_{\boldsymbol{m}}$ and σ only such that for every $f \in M^{1,p}(X, d, \boldsymbol{m})$ and $g \in D(f)$ we have*

(i) *If $0 < p < s$ and $p* = \frac{sp}{s-p}$, then $f \in L^{p*}(B_0)$ and*

$$\inf_{c\in\mathbb{R}} \Big(\fint_{B_0} |f - c|^{p*} d\boldsymbol{m} \Big)^{1/p*} \leq C r_o \Big(\fint_{\sigma B_0} g^p d\boldsymbol{m} \Big)^{1/p}.$$

(ii) *If $p = s$, then*

$$\fint_{B_0} \exp\Big(C_1 \frac{\boldsymbol{m}(\sigma B)^{\frac{1}{s}}}{r_o} \frac{|f - f_{B_0}|}{\|g\|_{L^s(\sigma B_0)}} \Big) d\boldsymbol{m} \leq C_2.$$

(iii) *If $p > s$, then there exists $\widetilde{f} = f$ $\boldsymbol{m}-$a.e. on σB_0 such that $\widetilde{f}$ is Hölder continuous on B_0 and*

$$|\widetilde{f}(x) - \widetilde{f}(y)| \leq C r_o^{\frac{s}{p}} d(x, y)^{1-\frac{s}{p}} \Big(\int_{\sigma B_0} g^p d\boldsymbol{m} \Big)^{1/p} \quad \text{for } x, y \in B_0.$$

2.4 Sobolev spaces $P^{1,p}$

The last approach to the Sobolev spaces in setting of metric measure spaces which we shall introduce is suggested by the following theorem.

Theorem 2.4.1 *For $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, the following conditions are equivalent:*

(i) $f \in W^{1,p}(\mathbb{R}^n)$,

(ii) *There exists $0 \leq g \in L^p(\mathbb{R}^n)$ such that*

$$\fint_B |f - f_B| dx \leq r \fint_B g dx$$

on every ball B of any radius r.

(iii) *There exists $0 \leq g \in L^p(\mathbb{R}^n)$, and $\sigma \geq 1$ such that*

$$\fint_B |f - f_B| dx \leq r \Big(\fint_{\sigma B} g^p dx \Big)^{1/p}$$

on every ball B of any radius r.

(iv) *There exists $0 \leq g \in L^p(\mathbb{R}^n)$, and $\sigma \geq 1$ such that*

$$|f(x) - f(y)| \leq |x - y| \big((\mathcal{M}_{\sigma|x-y|} g^p(x))^{1/p} + (\mathcal{M}_{\sigma|x-y|} g^p(y))^{1/p} \big) \quad \lambda_n\text{-a.e. on } \mathbb{R}^n.$$

Moreover, each of the inequalities at (ii)-(iv) *implies that*

$$|\nabla f| \le Cg \quad \lambda_n\text{-a.e. on } \mathbb{R}^n,$$

where C is a constant.

We assume that $(X, d, \boldsymbol{m})$ is metric measure space, $\boldsymbol{m}$ is doubling measure with doubling constant $C_{\boldsymbol{m}}$ and $s = \log_2 C_{\boldsymbol{m}}$ is the associated homogeneous dimension.

Definition 2.4.2 Fix $\sigma \ge 1$ and $0 < p < \infty$. We say that the pair (f, g), where $f \in L^1_{\text{loc}}(X)$ and $0 \le g \in L^p_{\text{loc}}(X)$, satisfies the *$p$-Poincaré inequality* if

$$⨍_B |f - f_B| d\boldsymbol{m} \le r \Big(⨍_{\sigma B} g^p d\boldsymbol{m} \Big)^{1/p} \text{ for every ball } B \subset X \text{ of radius } r. \tag{2.4}$$

Then we define $P^{1,p}_{\sigma,\text{loc}}(X, d, \boldsymbol{m})$ to be the class of all functions $f \in L^1_{\text{loc}}(X)$ for which there exists $0 \le g \in L^p_{\text{loc}}(X)$ so that the pair (f, g) satisfies the p-Poincaré inequality. We denote $P^{1,p}_{\text{loc}}(X, d, \boldsymbol{m}) = \bigcup_{\sigma \ge 1} P^{1,p}_{\sigma,\text{loc}}(X, d, \boldsymbol{m})$. Finally, $P^{1,p}(X, d, \boldsymbol{m})$ is the set of all functions $f \in L^1_{\text{loc}}(X)$ for which there exist $\sigma \ge 1$ and $0 \le g \in L^p(X)$ such that the p-Poincaré inequality (2.4) holds true for the pair (f, g).

As Theorems 2.4.3 and 2.4.4 show, implications (iii)⇒(iv) and (iv)⇒(iii) in Theorem 2.4.1, respectively, have direct generalizations to the environment of metric measure spaces.

Theorem 2.4.3 *If $f \in P^{1,p}_{\sigma,\text{loc}}(X, d, \boldsymbol{m})$ for some $p > 0$ and $\sigma \ge 1$, and $0 \le g \in L^p_{\text{loc}}(X)$ is the function for which the pair (f, g) satisfies the p-Poincaré inequality, then*

$$|f(x) - f(y)| \le C d(x, y) \Big(\big(\mathcal{M}_{2\sigma d(x,y)}(g^p)(x)\big)^{1/p} + \big(\mathcal{M}_{2\sigma d(x,y)}(g^p)(y)\big)^{1/p} \Big)$$

holds $\boldsymbol{m}-$a.e. on X.

Theorem 2.4.4 *If $\frac{s}{s+1} < p < \infty$ and $f \in L^1_{\text{loc}}(X)$, $g \in L^p_{\text{loc}}(X)$ are functions for which there exists a constant $\sigma \ge 1$ such that the inequality*

$$|f(x) - f(y)| \le d(x, y) \Big(\big(\mathcal{M}_{\sigma d(x,y)}(g^p)(x)\big)^{1/p} + \big(\mathcal{M}_{\sigma d(x,y)}(g^p)(y)\big)^{1/p} \Big)$$

holds $\boldsymbol{m}-$a.e. on X, then

$$⨍_B |f - f_B| d\boldsymbol{m} \le Cr \Big(⨍_{6\sigma B} g^p d\boldsymbol{m} \Big)^{1/p}$$

for every ball $B \subset X$ of radius r.

As well as for $M^{1,p}$ spaces, there exists a version of an embedding theorem for $P^{1,p}$ spaces.

Theorem 2.4.5 *Assume that the pair (f,g), where $f \in L^1_{\text{loc}}(X)$ and $0 \le g \in L^p_{\text{loc}}(X)$, satisfies the p-Poincaré inequality with some $0 < p < \infty$ and $\sigma \ge 1$.*

(i) *If $0 < p < s$, then for every $0 < h < \frac{sp}{s-p}$*

$$\inf_{c\in\mathbb{R}} \left(\fint_B |f-c|^h d\boldsymbol{m} \right)^{1/h} \le Cr \left(\fint_{6\sigma B} g^p d\boldsymbol{m} \right)^{1/p},$$

where $B \subset X$ is an arbitrary ball of radius r and C is a constant depending on p, h, $C_{\boldsymbol{m}}$ and σ. If, in addition, $g \in L^q_{\text{loc}}(X)$ for some $p < q < s$, then

$$\inf_{c\in\mathbb{R}} \left(\fint_B |f-c|^{q*} d\boldsymbol{m} \right)^{1/q*} \le Cr \left(\fint_{6\sigma B} g^q d\boldsymbol{m} \right)^{1/q},$$

where $q* = \frac{sq}{s-q}$ and $B \subset X$ is an arbitrary ball of radius r and C is a constant depending on p, q, $C_{\boldsymbol{m}}$ and σ.

(ii) *If $p = s$, then*

$$\fint_B \exp\left(\frac{C_1 \boldsymbol{m}(6\sigma B)^{1/s} |f - f_B|}{r\|g\|_{L^s(6\sigma B)}} \right) d\boldsymbol{m} \le C_2,$$

where $B \subset X$ is an arbitrary ball of radius r and C_1, C_2 are constants depending on p, $C_{\boldsymbol{m}}$ and σ.

(iii) *If $p > s$, then there exists $\widetilde{f} = f$ $\boldsymbol{m}$-a.e. on X satisfying*

$$|\widetilde{f}(x) - \widetilde{f}(y)| \le Cr^{\frac{s}{p}} d(x,y)^{1-\frac{s}{p}} \left(\fint_{6\sigma B} g^p d\boldsymbol{m} \right)^{1/p},$$

for all $x, y \in B$, where $B \subset X$ is an arbitrary ball of radius r and C is a constant depending on p, $C_{\boldsymbol{m}}$ and σ. Consequently, $\widetilde{f}$ is locally Hölder continuous on X.

2.5 Relationship of the spaces $N^{1,p}$, $M^{1,p}$ and $P^{1,p}$

Assume that $(X, d, \boldsymbol{m})$ is a metric measure space.

Theorem 2.5.1 *Let $1 \le p < \infty$. For each $f \in M^{1,p}(X,d,\boldsymbol{m})$ there exists a representative belonging to $\widetilde{N}^{1,p}(X,d,\boldsymbol{m})$ such that $2g$ is its p-weak upper gradient whenever $g \in D(f) \cap L^p(X)$. Thus* (due to Theorem 2.2.15) *we have*

$$M^{1,p}(X,d,\boldsymbol{m}) \subset N^{1,p}(X,d,\boldsymbol{m}) \quad \text{and} \quad \|f\|_{N^{1,p}} \le 2\|f\|_{M^{1,p}},$$

i.e. $M^{1,p}(X,d,\boldsymbol{m})$ is continuously embedded to $N^{1,p}(X,d,\boldsymbol{m})$.

According to Theorem 2.2.17 and Theorem 2.3.4, we obtain even an equality in the setting of $\mathbb{R}^n$ under certain circumstances.

Theorem 2.5.2 *If $1 < p < \infty$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain with strong local Lipschitz property or $\Omega = \mathbb{R}^n$, then*

$$M^{1,p}(\Omega, |\cdot|, \lambda_n) = N^{1,p}(\Omega, |\cdot|, \lambda_n).$$

Now, in addition, suppose that $\boldsymbol{m}$ is a doubling measure with doubling constant $C_{\boldsymbol{m}}$ and $s = \log_2 C_{\boldsymbol{m}}$ is the associated homogeneous dimension. Regarding $M^{1,p}$ and $P^{1,p}$ spaces, one can derive these inclusions:

Theorem 2.5.3 *For $p \geq \frac{s}{s+1}$ and $0 < q < p$ we have*

$$M^{1,p}(X, d, \boldsymbol{m}) \subset \big(P^{1,p}(X, d, \boldsymbol{m}) \cap L^p(X)\big) \subset M^{1,q}_{\text{loc}}(X, d, \boldsymbol{m}),$$

where $M^{1,q}_{\text{loc}}(X, d, \boldsymbol{m})$ is the space of all functions $f \in L^q_{\text{loc}}(X)$ for which $D(f) \cap L^q_{\text{loc}}(X) \neq \emptyset$ (to recall $D(f)$ see Definition 2.3.2).

The first inclusion follows from a stronger result which asserts that for $p \geq \frac{s}{s+1}$, any $\sigma > 1$, each $f \in M^{1,p}(X, d, \boldsymbol{m})$ and $g \in D(f)$ the inequality

$$\fint_B |f - f_B| d\boldsymbol{m} \leq Cr \Big(\fint_{\sigma B} g^p d\boldsymbol{m} \Big)^{1/p}$$

holds on every ball B of radius r with C depending on $C_{\boldsymbol{m}}$, p and σ. To get the second inclusion, we use Theorem 2.4.3 which implies

$$|f(x) - f(y)| \leq C d(x, y) \Big(\big(\mathcal{M}(g^p)(x)\big)^{1/p} + \big(\mathcal{M}(g^p)(y)\big)^{1/p} \Big) \quad \boldsymbol{m} - \text{a.e. on } X$$

for a pair (f, g) satisfying the p-Poincaré inequality, $p > 0$. Since $g^p \in L^1(X)$, the part (i) of Theorem 2.1.4 yields $(\mathcal{M}(g^p))^{1/p} \in$ weak-$L^p(X)$. This together with the fact that weak-$L^p(X) \subset L^q_{\text{loc}}(X)$ for all $0 < q < p$ gives the desired inclusion.

One important restriction of the class of all homogeneous spaces is formed by spaces supporting p-Poincaré inequalities. This subclass is at once specific enough to offer rich theory with strong outcomes and still sufficiently wide to cover Euclidean spaces as well as many non-Euclidean examples.

Definition 2.5.4 We say that a complete metric measure space $(X, d, \boldsymbol{m})$, equipped with a doubling measure, *supports a p-Poincaré inequality*, where $1 \leq p < \infty$, if there exist constants C_p and $\sigma \geq 1$ such that for every Borel function $f : X \to \mathbb{R}$ and every upper gradient $g : X \to [0, \infty]$ of f, the pair (f, g) satisfies p-Poincaré inequality

$$\fint_B |f - f_B| d\boldsymbol{m} \leq C_p r \Big(\fint_{\sigma B} g^p d\boldsymbol{m} \Big)^{1/p} \quad \text{on each ball } B \subset X \text{ of radius r.}$$

The Euclidean space supports p-Poincaré inequalities for all $p \geq 1$.

It turns out that on such spaces all three approaches to Sobolev spaces coincide.

Theorem 2.5.5 *If $1 < p < \infty$ and the space $(X, d, \boldsymbol{m})$ supports the q-Poincaré inequality for some $1 \leq q < p$, then*

$$M^{1,p}(X, d, \boldsymbol{m}) = P^{1,p}(X, d, \boldsymbol{m}) \cap L^p(X) = N^{1,p}(X, d, \boldsymbol{m}).$$

Remark 2.5.6 The latter equality holds also in the setting of space $(X, d, \boldsymbol{m})$ supporting p-Poincaré inequality, where $1 \leq p < \infty$.

As the concluding theorem shows, one of the advantages of the subclass of homogeneous spaces being discussed here is the fact that it guarantees certain features of Sobolev spaces built upon the elements of this subclass.

Theorem 2.5.7 *Let $1 < p < \infty$. If the space $(X, d, \boldsymbol{m})$ supports the p-Poincaré inequality, then $N^{1,p}(X, d, \boldsymbol{m})$ is reflexive. If, in addition, the space $(X, d, \boldsymbol{m})$ supports the q-Poincaré inequality for some $1 \leq q < p$, then $M^{1,p}(X, d, \boldsymbol{m})$ is reflexive too.*

Chapter 3

The main result

Notation. In what follows, $C(a_1, ..., a_k)$ always denotes a constant depending on the parameters $a_1, ..., a_k$ only, however, the value of the constant may change even within one string of (in)equalities. Next we write $L^p(X, d\mu)$, where X is a measure space and μ is a measure on X, instead of $L^p(X)$ in order to express that we integrate with respect to measure μ. We adopt the convention that $\int_a^b F(t)dt$ means $\int_a^b F(t)d\lambda(t)$, where λ is the Lebesgue measure on $\mathbb{R}$, $(a, b) \subset \mathbb{R}$ and F is Lebesgue measurable on $\mathbb{R}$. By a curve γ in Ω we always understand a nonconstant rectifiable arc-length parametrized curve, thus γ is always 1-Lipschitz.

Let $(X, |\cdot|, \boldsymbol{m})$ be a metric measure space and $\Omega \subset X$ be an open set with $\boldsymbol{m}(\Omega) < \infty$. We assume the existence of the following objects:

- constants $n \geq 1, A \geq 1, N \in \mathbb{N}$;
- a sequence $\{t_i\}_{i=0}^{\infty} \subset \mathbb{R}$ such that $0 = t_0 < t_1 < \ldots$; we denote $b = \lim_{i\to\infty} t_i$, $\varrho_i = t_i - t_{i-1}$ and $\varrho(t) = \varrho_i$ on $[t_{i-1}, t_i)$;
- a sequence $\{\Omega_i\}_{i=1}^{\infty}$ of open subsets of Ω;
- a linear operator $M : L^p_{\text{loc}}(\Omega, d\boldsymbol{m}) \to C([0, b))$;
- a sequence $\{\omega_i\}_{i=1}^{\infty}$ of real-valued functions on Ω (partition of unity).

We impose the following requirements on the listed objects:

(i)
$$A^{-1}\varrho_i \leq \varrho_{i-1} \leq A\varrho_i; \tag{3.1}$$

(ii)
$$A^{-1}\varrho_i^n \leq \boldsymbol{m}(\Omega_i) \leq A\varrho_i^n; \tag{3.2}$$

(iii)
$$\Omega_i \cap \Omega_j = \emptyset \quad \text{for } |i - j| > N; \tag{3.3}$$

(iv) For $f \in N^{1,p}(\Omega, |\cdot|, \boldsymbol{m})$, $1 < p < \infty$, the function Mf is always linear on $[t_{i-1}, t_i]$ and Mf, $(Mf)' \in L^p([0,b), d\nu)$. The measure ν is given on the σ - algebra of Lebesgue measurable sets in $[0,b)$ by $\nu(E) = \int_E \varrho^{n-1}(t)d\lambda(t)$, where E is a Lebesgue measurable subset of $[0,b)$ and λ is the 1-dimensional Lebesgue measure[1];

(v) For $f \in N^{1,p}(\Omega, |\cdot|, \boldsymbol{m})$, $1 < p < \infty$, and every[2] $g \in L^p(\Omega, d\boldsymbol{m})$ p-weak upper gradient of f

$$\frac{|Mf(t_i) - Mf(t_{i-1})|}{\varrho_i} + \left(\fint_{\Omega_i} |f(x) - Mf(t_i)|^p d\boldsymbol{m}(x) \right)^{1/p} \leq A \left(\fint_{\Omega_i} |g(x)|^p d\boldsymbol{m}(x) \right)^{1/p}.$$

In particular,

$$\left(\frac{|Mf(t_i) - Mf(t_{i-1})|}{\varrho_i} \right)^p \leq A^p \fint_{\Omega_i} |g(x)|^p d\boldsymbol{m}(x) \tag{3.4}$$

and

$$\fint_{\Omega_i} |f(x) - Mf(t_i)|^p d\boldsymbol{m}(x) \leq A^p \fint_{\Omega_i} |g(x)|^p d\boldsymbol{m}(x); \tag{3.5}$$

(vi)

$$0 \leq \omega_i \leq 1 \quad \text{on } \Omega, \tag{3.6}$$

$$\omega_i = 0 \quad \text{on } \Omega \setminus \Omega_i, \tag{3.7}$$

$$\sum_i \omega_i \equiv 1 \quad \text{on } \Omega, \tag{3.8}$$

$$\omega_i \text{ is Lipschitz continuous on } \Omega \text{ with constant } \operatorname{lip}(\omega_i) \leq \frac{A}{\varrho_i}. \tag{3.9}$$

Now we are in a position to formulate our main result.

Theorem 3.0.8 *If the above conditions are satisfied, then the following two assertions are equivalent:*

(I) *There exists $C = C(A, N, n, p)$ such that*

$$\inf_{a \in \mathbb{R}} \|f - a\|_{L^p(\Omega, d\boldsymbol{m})} \leq C \|g\|_{L^p(\Omega, d\boldsymbol{m})}$$

for every $f \in N^{1,p}(\Omega, |\cdot|, \boldsymbol{m})$ and each p-weak upper gradient $g \in L^p(\Omega, d\boldsymbol{m})$ of f;

(II) *There exists $C = C(A, N, n, p)$ such that*

$$\inf_{a \in \mathbb{R}} \|F - a\|_{L^p([0,b), d\nu)} \leq C \|F'\|_{L^p([0,b), d\nu)}$$

for every continuous function $F : [0, b) \to \mathbb{R}$ such that $F|_{[t_{i-1}, t_i]}$ is linear for each $i \in \mathbb{N}$ and F, $F' \in L^p([0,b), d\nu)$.

[1] Note that for Mf, as a continuous function linear on $[t_{i-1}, t_i]$, $(Mf)'$ exists $\lambda-$a.e., therefore also $\nu-$a.e. on $(0, b)$.

[2] According to Theorem 2.2.20, the condition "every $g \in L^p(\Omega, d\boldsymbol{m})$ p-weak upper gradient of f" can be replaced by "the least p-weak upper gradient $g_f \in L^p(\Omega, d\boldsymbol{m})$ of f".

Before the proof of the theorem itself, let us state some general estimates which turn out to be useful later.
First, we present several well-known results from the theory of Lebesgue spaces.

- Hölder's inequality:

$$\|fg\|_{L^1(\Omega,d\boldsymbol{m})} \leq \|f\|_{L^p(\Omega,d\boldsymbol{m})}\|g\|_{L^{\frac{p}{p-1}}(\Omega,d\boldsymbol{m})} \tag{3.10}$$

  for $1 < p < \infty$, $f \in L^p(\Omega, d\boldsymbol{m})$, $g \in L^{\frac{p}{p-1}}(\Omega, d\boldsymbol{m})$.

- Minkowski's inequality:

$$\|f+g\|_{L^p(\Omega,d\boldsymbol{m})} \leq \|f\|_{L^p(\Omega,d\boldsymbol{m})} + \|g\|_{L^p(\Omega,d\boldsymbol{m})} \tag{3.11}$$

  for $1 < p < \infty$ and $f, g \in L^p(\Omega, d\boldsymbol{m})$.

Both have also discrete versions:

- Hölder's inequality for series:

$$\sum_{i=1}^{\infty} a_i b_i \leq \Big(\sum_{i=1}^{\infty} a_i^p\Big)^{1/p}\Big(\sum_{i=1}^{\infty} b_i^{\frac{p}{p-1}}\Big)^{\frac{p-1}{p}} \tag{3.12}$$

  for $1 < p < \infty$, $\{a_i\}_{i=1}^{\infty} \subset \mathbb{R}_0^+$, $\{b_i\}_{i=1}^{\infty} \subset \mathbb{R}_0^+$.

- Minkowski's inequality for series:

$$\Big(\sum_{i=1}^{\infty}(a_i+b_i)^p\Big)^{1/p} \leq \Big(\sum_{i=1}^{\infty} a_i^p\Big)^{1/p} + \Big(\sum_{i=1}^{\infty} b_i^p\Big)^{1/p} \tag{3.13}$$

  for $1 < p < \infty$, $\{a_i\}_{i=1}^{\infty} \subset \mathbb{R}_0^+$, $\{b_i\}_{i=1}^{\infty} \subset \mathbb{R}_0^+$.

Among elementary inequalities in analysis belongs:

$$(a+b)^p \leq 2^{p-1}(a^p+b^p) \quad \text{for } a \geq 0,\ b \geq 0,\ 1 \leq p < \infty, \tag{3.14}$$

implied by the convexity of the p-th power restricted on nonnegative real numbers.

Proof of Theorem 3.0.8

(II)⇒ (I) Let (II) be satisfied. Take $f \in N^{1,p}(\Omega, |\cdot|, \boldsymbol{m})$ and $g \in L^p(\Omega, d\boldsymbol{m})$ an arbitrary p-weak upper gradient of f. Then for any $a \in \mathbb{R}$ we have

$$\begin{aligned}
\|f-a\|_{L^p(\Omega,d\boldsymbol{m})} &= \Big(\int_\Omega |f(x)-a|^p d\boldsymbol{m}(x)\Big)^{1/p}\\
&\le \Big(\sum_{i=1}^\infty \int_{\Omega_i} |f(x)-a|^p d\boldsymbol{m}(x)\Big)^{1/p}\\
&= \Big(\sum_{i=1}^\infty \int_{\Omega_i} |f(x)-Mf(t_i)+Mf(t_i)-a|^p d\boldsymbol{m}(x)\Big)^{1/p}\\
&\le \Big(\sum_{i=1}^\infty \int_{\Omega_i} \big(|f(x)-Mf(t_i)|+|Mf(t_i)-a|\big)^p d\boldsymbol{m}(x)\Big)^{1/p}\\
&\le \Bigg(\sum_{i=1}^\infty \bigg(\Big(\int_{\Omega_i} |f(x)-Mf(t_i)|^p d\boldsymbol{m}(x)\Big)^{1/p}\\
&\quad + \Big(\int_{\Omega_i} |Mf(t_i)-a|^p d\boldsymbol{m}(x)\Big)^{1/p}\bigg)^p\Bigg)^{1/p},
\end{aligned}$$

by Minkowski's inequality (3.11). Now, applying the discrete Minkowski's inequality (3.13) and the requirement (3.2) above gives

$$\begin{aligned}
&\|f-a\|_{L^p(\Omega,d\boldsymbol{m})}\\
&\quad\le \Big(\sum_{i=1}^\infty \int_{\Omega_i} |f(x)-Mf(t_i)|^p d\boldsymbol{m}(x)\Big)^{1/p} + \Big(\sum_{i=1}^\infty \int_{\Omega_i} |Mf(t_i)-a|^p d\boldsymbol{m}(x)\Big)^{1/p}\\
&\quad\le \Big(\sum_{i=1}^\infty \frac{A\varrho_i^n}{\boldsymbol{m}(\Omega_i)} \int_{\Omega_i} |f(x)-Mf(t_i)|^p d\boldsymbol{m}(x)\Big)^{1/p}\\
&\qquad + \Big(\sum_{i=1}^\infty \frac{A\varrho_i^n}{\boldsymbol{m}(\Omega_i)} \int_{\Omega_i} |Mf(t_i)-a|^p d\boldsymbol{m}(x)\Big)^{1/p}\\
&\quad= \Big(\sum_{i=1}^\infty A\varrho_i^n \fint_{\Omega_i} |f(x)-Mf(t_i)|^p d\boldsymbol{m}(x)\Big)^{1/p} + \Big(\sum_{i=1}^\infty A\varrho_i^n |Mf(t_i)-a|^p\Big)^{1/p}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\inf_{a\in\mathbb{R}} \|f-a\|_{L^p(\Omega,d\boldsymbol{m})} \le A^{1/p}&\Big(\sum_{i=1}^\infty \varrho_i^n \fint_{\Omega_i} |f(x)-Mf(t_i)|^p d\boldsymbol{m}(x)\Big)^{1/p}\\
&+ A^{1/p} \inf_{a\in\mathbb{R}} \Big(\sum_{i=1}^\infty \varrho_i^n |Mf(t_i)-a|^p\Big)^{1/p}. \qquad (3.15)
\end{aligned}$$

Concerning the first summand in (3.15), the conditions (3.5), (3.2) and (3.3) yield

$$\begin{aligned}\Big(\sum_{i=1}^{\infty}\varrho_i^n \fint_{\Omega_i}|f(x)-Mf(t_i)|^p d\boldsymbol{m}(x)\Big)^{1/p} &\le \Big(\sum_{i=1}^{\infty}\varrho_i^n A^p \fint_{\Omega_i}|g(x)|^p d\boldsymbol{m}(x)\Big)^{1/p}\\ &\le \Big(\sum_{i=1}^{\infty}\frac{\varrho_i^n A^p}{A^{-1}\varrho_i^n}\int_{\Omega_i}|g(x)|^p d\boldsymbol{m}(x)\Big)^{1/p}\\ &= A^{\frac{p+1}{p}}\Big(\sum_{i=1}^{\infty}\int_{\Omega_i}|g(x)|^p d\boldsymbol{m}(x)\Big)^{1/p}\\ &\le A^{\frac{p+1}{p}}\Big((2N+1)\int_{\Omega}|g(x)|^p d\boldsymbol{m}(x)\Big)^{1/p}\\ &= A^{\frac{p+1}{p}}(2N+1)^{1/p}\|g\|_{L^p(\Omega,d\boldsymbol{m})}.\end{aligned}$$

To get an appropriate estimate for the second summand in (3.15), we shall need the following observation[3]:

$$\varrho_i^n|Mf(t_i)|^p \le 2^{2p-1}A^n\int_{t_{i-1}}^{t_{i+1}}\varrho(t)^{n-1}|Mf(t)|^p dt \quad \text{for } i\in\mathbb{N}. \tag{3.16}$$

In case when $|Mf(t_{i-1})| \ge |Mf(t_i)|$, or $Mf(t)\ge 0$ on $[t_{i-1},t_i]$, or $Mf(t)\le 0$ on $[t_{i-1},t_i]$, the same is satisfied by $\alpha Mf(t)$ for any constant $\alpha>0$ and the observation is clear then. Indeed,

$$\varrho_i\alpha|Mf(t_i)| \le 2\int_{t_{i-1}}^{t_i}\alpha|Mf(t)|dt$$

holds for every $\alpha>0$. Whence, by using for $\alpha=\varrho_i^{\frac{n-p}{p}}$ and applying Hölder's inequality (3.10), we obtain

$$\begin{aligned}\varrho_i^n|Mf(t_i)|^p &= \big(\varrho_i\varrho_i^{\frac{n-p}{p}}|Mf(t_i)|\big)^p\\ &\le \Big(2\int_{t_{i-1}}^{t_i}\varrho_i^{\frac{n-p}{p}}|Mf(t)|dt\Big)^p = \Big(2\int_{t_{i-1}}^{t_i}\varrho_i^{\frac{n-1+1-p}{p}}|Mf(t)|dt\Big)^p\\ &\le 2^p\int_{t_{i-1}}^{t_i}\varrho_i^{n-1}|Mf(t)|^p dt\Big(\int_{t_{i-1}}^{t_i}\varrho_i^{-1}dt\Big)^{p-1}\\ &= 2^p\int_{t_{i-1}}^{t_i}\varrho(t)^{n-1}|Mf(t)|^p dt\\ &\le 2^{2p-1}A^n\int_{t_{i-1}}^{t_{i+1}}\varrho(t)^{n-1}|Mf(t)|^p dt.\end{aligned}$$

Provided that none of the preceding situations happens, that is when $|Mf(t_{i-1})|<|Mf(t_i)|$ and $|Mf(t)|$ is not linear on $[t_{i-1},t_i]$, denote with ξ_i the point satisfying $\xi_i\in[t_{i-1},t_i]$ and $Mf(\xi_i)=0$ (note that under our circumstances such point exists and is unique). Again, the function $\alpha Mf(t)$, where $\alpha>0$ fixed, inherits all just mentioned features of the function $Mf(t)$. Therefore,

$$(t_i-\xi_i)\alpha|Mf(t_i)| \le 2\int_{t_{i-1}}^{t_i}\alpha|Mf(t)|dt. \tag{3.17}$$

[3]Actually, this claim holds not only for the elements of the range of operator M, as we formulate it, but also for any continuous function $F:[0,b)\to\mathbb{R}$ such that $F|_{[t_{i-1},t_i]}$ is linear for each i and $F,F'\in L^p([0,b),d\nu)$.

Since $|Mf(t_{i-1})| < |Mf(t_i)|$, certainly $|\xi_i - t_{i-1}| \leq \frac{\varrho_i}{2}$. Moreover, inequality

$$\frac{\varrho_{i+1}}{2}\alpha|Mf(t_i)| \leq 2\int_{t_i}^{t_{i+1}} \alpha|Mf(t)|dt$$

holds always for $Mf(t)$ and $\alpha > 0$, independent on the behaviour of the function $Mf(t)$ on the interval $[t_{i-1}, t_i]$. Thus for any $\beta \in \mathbb{R}$,

$$\begin{aligned}(\xi_i - t_{i-1})\varrho_i^{\beta-1}|Mf(t_i)| &\leq \frac{\varrho_i^{\beta}}{2}|Mf(t_i)| \\ &\leq \frac{A^{\beta}\varrho_{i+1}^{\beta}}{2}|Mf(t_i)| \\ &\leq 2A^{\beta}\int_{t_i}^{t_{i+1}} \varrho_{i+1}^{\beta-1}|Mf(t)|dt \\ &= 2A^{\beta}\int_{t_i}^{t_{i+1}} \varrho(t)^{\beta-1}|Mf(t)|dt.\end{aligned}$$

In order to check the second inequality, we refer to (3.1). This result together with (3.17) for $\alpha = \varrho_i^{\beta-1}$ yields

$$\begin{aligned}\varrho_i^{\beta}|Mf(t_i)| &= (t_i - \xi_i + \xi_i - t_{i-1})\varrho_i^{\beta-1}|Mf(t_i)| \\ &\leq 2\int_{t_{i-1}}^{t_i} \varrho_i^{\beta-1}|Mf(t)|dt + 2A^{\beta}\int_{t_i}^{t_{i+1}} \varrho_{i+1}^{\beta-1}|Mf(t)|dt \\ &\leq 2A^{\beta}\int_{t_{i-1}}^{t_{i+1}} \varrho(t)^{\beta-1}|Mf(t)|dt.\end{aligned}$$

In particular, when applied to $\beta = \frac{n}{p}$, this gives

$$\begin{aligned}\varrho_i^{n}|Mf(t_i)|^p &= \big(\varrho_i^{\frac{n}{p}}|Mf(t_i)|\big)^p \\ &\leq \Big(2A^{\frac{n}{p}}\int_{t_{i-1}}^{t_{i+1}} \varrho(t)^{\frac{n-p}{p}}|Mf(t)|dt\Big)^p.\end{aligned}$$

Similarly to the previous case, the final estimate is implied by Hölder's inequality ,

$$\begin{aligned}\varrho_i^{n}|Mf(t_i)|^p &\leq 2^pA^n\Big(\int_{t_{i-1}}^{t_{i+1}} \varrho(t)^{\frac{n-1+1-p}{p}}|Mf(t)|dt\Big)^p \\ &\leq 2^pA^n\int_{t_{i-1}}^{t_{i+1}} \varrho(t)^{n-1}|Mf(t)|^pdt\Big(\int_{t_{i-1}}^{t_{i+1}} \varrho(t)^{-1}dt\Big)^{p-1} \\ &= 2^{2p-1}A^n\int_{t_{i-1}}^{t_{i+1}} \varrho(t)^{n-1}|Mf(t)|^pdt.\end{aligned}$$

Now, due to the observation (3.16), we easily arrive at

$$\begin{aligned}\sum_{i=1}^{\infty}\varrho_i^{n}|Mf(t_i)|^p &\leq 2^{2p-1}A^n\sum_{i=1}^{\infty}\int_{t_{i-1}}^{t_{i+1}} \varrho(t)^{n-1}|Mf(t)|^pdt \\ &\leq 4^pA^n\int_0^b \varrho(t)^{n-1}|Mf(t)|^pdt. \qquad (3.18)\end{aligned}$$

Finally, we are in a position to derive the expected estimate for the second summand in (3.15). To this end, we shall use (II). Note that when $Mf(t)$ is continuous and linear on each $[t_{i-1},t_i]$, then so is $Mf(t)-a$ for any $a\in\mathbb{R}$. Consequently, the claims just presented hold for $Mf(t)-a$ in the same way as for $Mf(t)$. Thus, we have

$$\begin{aligned}
\inf_{a\in\mathbb{R}}\Big(\sum_{i=1}^{\infty}\varrho_i^n|Mf(t_i)-a|^p\Big)^{1/p} &\le \inf_{a\in\mathbb{R}}\Big(4^pA^n\int_0^b\varrho(t)^{n-1}|Mf(t)-a|^p dt\Big)^{1/p}\\
&\le C(A,N,n,p)\Big(\int_0^b\varrho(t)^{n-1}|(Mf)'(t)|^p dt\Big)^{1/p}\\
&= C(A,N,n,p)\Big(\sum_{i=1}^{\infty}\int_{t_{i-1}}^{t_i}\varrho(t)^{n-1}|(Mf)'(t)|^p dt\Big)^{1/p}.
\end{aligned}$$

After rewriting the derivative into the form of a function which is constant on each interval $[t_{i-1},t_i]$ and a subsequent use of the conditions (3.4), (3.2) and (3.3), respectively, we obtain

$$\begin{aligned}
\inf_{a\in\mathbb{R}}\Big(\sum_{i=1}^{\infty}\varrho_i^n|Mf(t_i)-a|^p\Big)^{1/p} &\le C(A,N,n,p)\Big(\sum_{i=1}^{\infty}\varrho_i\Big(\varrho_i^{n-1}\frac{|Mf(t_i)-Mf(t_{i-1})|^p}{\varrho_i^p}\Big)\Big)^{1/p}\\
&\le C(A,N,n,p)\Big(\sum_{i=1}^{\infty}\varrho_i^nA^p\fint_{\Omega_i}|g(x)|^p d\boldsymbol{m}(x)\Big)^{1/p}\\
&\le C(A,N,n,p)\Big(\sum_{i=1}^{\infty}\frac{\varrho_i^n}{A^{-1}\varrho_i^n}\int_{\Omega_i}|g(x)|^p d\boldsymbol{m}(x)\Big)^{1/p}\\
&= C(A,N,n,p)\Big(\sum_{i=1}^{\infty}\int_{\Omega_i}|g(x)|^p d\boldsymbol{m}(x)\Big)^{1/p}\\
&\le C(A,N,n,p)\Big((2N+1)\int_{\Omega}|g(x)|^p d\boldsymbol{m}(x)\Big)^{1/p}\\
&= C(A,N,n,p)\|g\|_{L^p(\Omega,d\boldsymbol{m})}.
\end{aligned}$$

In conclusion, an application of the estimates carried out for the summands in (3.15) shows that

$$\inf_{a\in\mathbb{R}}\|f-a\|_{L^p(\Omega,d\boldsymbol{m})}\le C(A,N,n,p)\|g\|_{L^p(\Omega,d\boldsymbol{m})}$$

as desired.

(I)⇒(II) Conversely, suppose that the statement (I) holds. Take a real-valued continuous function $F\in L^p([0,b),d\nu)$ linear on $[t_{i-1},t_i]$ for each $i\in\mathbb{N}$, for which also $F'\in L^p([0,b),d\nu)$. Our intention is to establish the inequality in (II) for a given F. Note that the just described function F is uniquely determined by a sequence $\{F(t_i)\}_{i=0}^{\infty}\subset\mathbb{R}$ (throughout the rest of this section we abbreviate $F_i=F(t_i)$).

Define $f(x):=\sum_{i=1}^{\infty}F_{i-1}\omega_i(x)$ for $x\in\Omega$. We want f to serve us to reach our goal. To this end, we must first show that f is a function on Ω satisfying the assumptions of statement (I) and afterwards find an appropriate relation between f and F leading to estimates beneficial for our aim.

To begin with, $f : \Omega \to \mathbb{R}$ is correctly defined. Indeed, ω_i for each $i \in \mathbb{N}$ is real-valued function on Ω and $\{F_i\}_{i=0}^{\infty} \subset \mathbb{R}$. Furthermore, for all $x \in \Omega$ we have $\omega_i(x) \neq 0$ for no more than 2N+1 indices i, therefore the sum is finite and $f(x)$ is well defined.

The function $\omega_i(x)$ is continuous on Ω for each $i \in \mathbb{N}$, thus $\sum_{i=1}^{l} F_{i-1}\omega_i(x)$ is continuous for each $l \in \mathbb{N}$ as well. That is why $f(x)$ as the pointwise limit of the sequence of functions $\{\sum_{i=1}^{l} F_{i-1}\omega_i(x)\}_{l=1}^{\infty}$ is of the first Baire class and so Borel, hence $\boldsymbol{m}-$measurable.

Thanks to the fact that $F \in L^p([0,b), d\nu)$, we can be sure that $\int_\Omega |f(x)|^p d\boldsymbol{m}(x) < \infty$, as will be verified immediately.

We shall now introduce certain notation and present an observation, whose result will be used several times in the remainder of the proof.
Notation. We adopt a convention that for each $i \in \mathbb{N}$, M_i denotes the set $\{i - N, \dots, i, \dots, i + N\} \subset \mathbb{N}$, eventually the set $\{1, \dots, i, \dots, i + N\} \subset \mathbb{N}$ when $i - N \leq 0$, where N is a constant defined in the first item of the list at the very beginning of this chapter. Next, for each $i \in \mathbb{N}$, Q_i denotes the set $\{i - 2N, \dots, i, \dots, i + 2N\} \subset \mathbb{N}$, eventually the set $\{1, \dots, i, \dots, i + 2N\} \subset \mathbb{N}$ when $i - 2N \leq 0$, where N is a constant defined in the first item of the list at the very beginning of this chapter.
Our observation is based on the condition (3.1) and it reads as:

$$A^{-N}\varrho_i \leq \varrho_{i_o} \leq A^N\varrho_i \quad \text{for } i_0 \in \mathbb{N} \text{ fixed and any } i \in M_{i_o}. \tag{3.19}$$

Consequently,

$$A^{-2N}\varrho_i \leq \varrho_j \leq A^{2N}\varrho_i \quad \text{for } i_0 \in \mathbb{N} \text{ fixed and any } i, j \in M_{i_o}. \tag{3.20}$$

Now, we come back to problem, whether $\int_\Omega |f(x)|^p d\boldsymbol{m}(x) < \infty$ is true.

$$\begin{aligned}
\int_\Omega |f(x)|^p d\boldsymbol{m}(x) &\leq \sum_{i=1}^{\infty} \int_{\Omega_i} |f(x)|^p d\boldsymbol{m}(x) = \sum_{i=1}^{\infty} \int_{\Omega_i} \Big| \sum_{j \in M_i} F_{j-1}\omega_j(x) \Big|^p d\boldsymbol{m}(x) \\
&\leq \sum_{i=1}^{\infty} \int_{\Omega_i} \Big(\sum_{j \in M_i} |F_{j-1}|\omega_j(x) \Big)^p d\boldsymbol{m}(x) \\
&\leq \sum_{i=1}^{\infty} \int_{\Omega_i} \Big(\sum_{j \in M_i} |F_{j-1}| \Big)^p d\boldsymbol{m}(x) \\
&= \sum_{i=1}^{\infty} \Big(\sum_{j \in M_i} |F_{j-1}| \Big)^p \boldsymbol{m}(\Omega_i),
\end{aligned}$$

where we used the definition of f and the properties of ω_i, namely (3.7) and (3.6). Because the inner sum is taken over a finite set of indices whose upper bound for volume depends on N only, we can, if roughly speaking, get the p-th power inside the sum by applying (3.14) as many times as needed (more precisely, finitely many times dependent on N). As the whole series is nonnegative, we can rearrange its elements without change in terms of convergence. Thus,

$$\begin{aligned}\int_\Omega |f(x)|^p d\boldsymbol{m}(x) &\le (2^{p-1})^{2N} \sum_{i=1}^{\infty} \sum_{j\in M_i} |F_{j-1}|^p \boldsymbol{m}(\Omega_i)\\ &= C(N,p) \sum_{i=1}^{\infty} \sum_{j\in M_i} \boldsymbol{m}(\Omega_j)|F_{i-1}|^p.\end{aligned}$$

Next comes a string of estimates following from (3.2), (3.19), (3.18) and the assumption on F, respectively. We arrive at

$$\begin{aligned}\int_\Omega |f(x)|^p d\boldsymbol{m}(x) &\le C(N,p) \sum_{i=1}^{\infty} \sum_{j\in M_i} A\varrho_j^n |F_{i-1}|^p\\ &\le C(N,p) \sum_{i=1}^{\infty} \sum_{j\in M_i} A(A^N\varrho_i)^n |F_{i-1}|^p\\ &= C(A,N,n,p) \sum_{i=1}^{\infty} \varrho_i^n |F_{i-1}|^p\\ &\le C(A,N,n,p) \int_0^b |F(t)|^p \varrho(t)^{n-1} dt\\ &= C(A,N,n,p) \|F\|^p_{L^p([0,b),d\nu)} < \infty.\end{aligned}$$

The list of the claims imposed on functions considered in (I) of the theorem completes the requirement for the existence of a p-weak upper gradient belonging to $L^p(\Omega, d\boldsymbol{m})$. We assert that also this property is possessed by our function f and that for such p-weak upper gradient of f we can take a function

$$\begin{aligned}g(x) &= A^{2N+1}(2N+2) \sum_{k=1}^{\infty} \Big(\sum_{j=1}^{\infty} \frac{|F_{k-1} - F_{j-1}|}{\varrho_j} \omega_j(x) \Big) \chi_{\bigcup\limits_{l\in M_k} \Omega_l}(x)\\ &= \kappa \sum_{k=1}^{\infty} \Big(\sum_{j=1}^{\infty} \frac{|F_{k-1} - F_{j-1}|}{\varrho_j} \omega_j(x) \Big) \chi_{\bigcup\limits_{l\in M_k} \Omega_l}(x),\end{aligned}$$

say. Actually, we shall show that g is even an upper gradient of f. Above all, g is a nonnegative and Borel function on Ω. Indeed, for each $k \in \mathbb{N}$, $\sum_{j=1}^{\infty} \frac{|F_{k-1}-F_{j-1}|}{\varrho_j}\omega_j(x)$ as a pointwise limit of continuous functions is Borel and $\chi_{\bigcup\limits_{l\in M_k} \Omega_l}(x)$ for $\bigcup\limits_{l\in M_k} \Omega_l$ open (so Borel) is Borel too. Thus any finite sum $\sum_{k=1}^{l} \Big(\sum_{j=1}^{\infty} \frac{|F_{k-1}-F_{j-1}|}{\varrho_j}\omega_j(x) \Big) \chi_{\bigcup\limits_{l\in M_k} \Omega_l}(x)$ is Borel. Then so is also the pointwise limit for l tending to ∞, i.e. the function g. Next, we need g to belong to $L^p(\Omega, d\boldsymbol{m})$.

To this end, we derive here an estimate which will be helpful in the final part of the proof as well.

We have

$$
\begin{aligned}
\|g\|^p_{L^p(\Omega,d\boldsymbol{m})} &= \int_\Omega |g(x)|^p d\boldsymbol{m}(x) = \int_\Omega \Big|\kappa \sum_{k=1}^\infty \Big(\sum_{j=1}^\infty \frac{|F_k - F_j|}{\varrho_j}\omega_j(x)\Big)\chi_{\Omega_k}(x)\Big|^p d\boldsymbol{m}(x) \\
&\le \sum_{i=1}^\infty \int_{\Omega_i} \Big(\kappa \sum_{k=1}^\infty \Big(\sum_{j=1}^\infty \frac{|F_k - F_j|}{\varrho_j}\omega_j(x)\Big)\chi_{\Omega_k}(x)\Big)^p d\boldsymbol{m}(x) \\
&\le \kappa^p \sum_{i=1}^\infty \int_{\Omega_i} \Big(\sum_{k\in Q_i}\sum_{j\in M_i} \frac{|F_{k-1} - F_{j-1}|}{\varrho_j}\omega_j(x)\Big)^p d\boldsymbol{m}(x) \\
&\le \kappa^p \sum_{i=1}^\infty \int_{\Omega_i} \Big(\sum_{k\in Q_i}\sum_{j\in M_i} \frac{|F_{k-1} - F_{j-1}|}{A^{-N}\varrho_i}\omega_j(x)\Big)^p d\boldsymbol{m}(x) \\
&\le \kappa^p A^N \sum_{i=1}^\infty \int_{\Omega_i} \bigg((2N+1) \max_{(k,j)\in Q_i\times M_i}\Big\{\frac{|F_{k-1} - F_{j-1}|}{\varrho_i}\Big\}\sum_{j\in M_i}\omega_j(x)\bigg)^p d\boldsymbol{m}(x) \\
&= \kappa^p A^N (2N+1)^p \sum_{i=1}^\infty \boldsymbol{m}(\Omega_i)\Big(\max_{(k,j)\in Q_i\times M_i}\Big\{\frac{|F_{k-1} - F_{j-1}|}{\varrho_i}\Big\}\Big)^p \\
&\le \kappa^p A^{N+1} (2N+1)^p \sum_{i=1}^\infty \varrho_i^n\Big(\max_{(k,j)\in Q_i\times M_i}\Big\{\frac{|F_{k-1} - F_{j-1}|}{\varrho_i}\Big\}\Big)^p,
\end{aligned}
$$

where we used (3.7), (3.19), (3.8) and (3.2). By the triangle inequality, the principle (3.14) iterated sufficiently many times and observation (3.19), we get

$$
\begin{aligned}
\|g\|^p_{L^p(\Omega,d\boldsymbol{m})} &\le \kappa^p A^{N+1}(2N+1)^p \sum_{i=1}^\infty \varrho_i^n \Big(\sum_{j\in Q_i}\frac{|F_j - F_{j-1}|}{\varrho_i}\Big)^p \\
&\le \kappa^p A^{N+1}(2N+1)^p (2^{p-1})^{4N} \sum_{i=1}^\infty \varrho_i^n \sum_{j\in Q_i}\frac{|F_j - F_{j-1}|^p}{\varrho_i^p} \\
&\le \kappa^p A^{N+1+2Np}(2N+1)^p (2^{p-1})^{4N} \sum_{i=1}^\infty \varrho_i^n \sum_{j\in Q_i}\frac{|F_j - F_{j-1}|^p}{\varrho_j^p}.
\end{aligned}
$$

The nonnegativity of the series allows us to rearrange its terms without change in the convergence. After this is done, we apply observation (3.19) one more time.

Finally, the form of the derivative of the function F yields

$$
\begin{aligned}
\|g\|^p_{L^p(\Omega,d\boldsymbol{m})} &\leq C(A,N,n,p)\sum_{j=1}^{\infty}\left(\frac{|F_j-F_{j-1}|^p}{\varrho_j^p}\sum_{i\in Q_j}\varrho_i^n\right)\\
&\leq C(A,N,n,p)\sum_{j=1}^{\infty}\left(\frac{|F_j-F_{j-1}|^p}{\varrho_j^p}\sum_{i\in Q_j}(A^N\varrho_j)^n\right)\\
&= C(A,N,n,p)\sum_{j=1}^{\infty}\frac{|F_j-F_{j-1}|^p}{\varrho_j^p}\varrho_j^n\\
&= C(A,N,n,p)\sum_{j=1}^{\infty}\int_{t_{j-1}}^{t_j}|F'(t)|^p\varrho_j^{n-1}dt\\
&= C(A,N,n,p)\int_0^b|F'(t)|^p\varrho(t)^{n-1}\,dt\\
&= C(A,N,n,p)\|F'\|^p_{L^p([0,b),d\nu)}. \qquad (3.21)
\end{aligned}
$$

Now we are left with the proof of inequality (2.1) for functions f and g defined in this section and all curves in Ω. To begin, let us make the following consideration, which helps to simplify our further work.
Suppose that (2.1) is satisfied for all curves γ such that $\langle\gamma\rangle\subset\Omega_i$ for some $i\in\mathbb{N}$, i.e. g is an upper gradient of f on Ω_i for each $i\in\Omega$. For an arbitrary curve γ in Ω choose a finite division s_j of interval $[0,l(\gamma)]$ satisfying that $\gamma([s_{j-1},s_j])\subset\Omega_i$ for some $i\in\mathbb{N}$[4]. Then

$$
|(f\circ\gamma)(l(\gamma))-(f\circ\gamma)(0)|\leq\sum_j|(f\circ\gamma)(s_j)-(f\circ\gamma)(s_{j-1})|\leq\sum_j\int_{s_{j-1}}^{s_j}(g\circ\gamma)(t)dt=\int_{\langle\gamma\rangle}g.
$$

This shows that it is sufficient to focus on the verification of inequality (2.1) for curves lying within Ω_{i_0} for $i_0\in\mathbb{N}$ fixed. According to (3.9), functions ω_i, for every $i\in\mathbb{N}$, are Lipschitz continuous on Ω. Due to (3.3), the sum on the right hand side of $f(x):=\sum_{i=1}^{\infty}F_i\omega_i(x)$ is finite on Ω_i for each $i\in\mathbb{N}$, therefore f is locally Lipschitz continuous on Ω, so Lipschitz continuous on $K\subset\Omega$, whenever K is a compact set. In particular, f is Lipschitz continuous on every curve $\langle\gamma\rangle$ in Ω. This implies that $(f\circ\gamma)(t)$ is Lipschitz continuous on $[0,l(\gamma)]$, hence differentiable $\lambda-$a.e. on $(0,l(\gamma))$ and $(f\circ\gamma)(l(\gamma))-(f\circ\gamma)(0)=\int_0^{l(\gamma)}(f\circ\gamma)'(t)dt$. Thus, if $|(f\circ\gamma)'(t)|\leq(g\circ\gamma)(t)$ $\lambda-$a.e. on $(0,l(\gamma))$, then

$$
|(f\circ\gamma)(l(\gamma))-(f\circ\gamma)(0)|=\int_0^{l(\gamma)}|(f\circ\gamma)'(t)|dt\leq\int_0^{l(\gamma)}(g\circ\gamma)(t)dt=\int_{\langle\gamma\rangle}g.
$$

Altogether, the above argument reduces our problem to the proof of the inequality $|(f\circ\gamma)'(t)|\leq(g\circ\gamma)(t)$ λ-a.e. on $(0,l(\gamma))$ for $\gamma:[0,l(\gamma)]\to\Omega_{i_0}$ for fixed $i_0\in\mathbb{N}$.

[4]The existence of such partition is a consequence of the compactness of $[0,l(\gamma)]$.

Referring only to the conditions (3.7), (3.8), (3.9) and the 1-Lipschitz continuity of γ, with series of elementary operations, we easily obtain (in)equalities holding λ-a.e. on $(0, l(\gamma))$

$$\begin{aligned}
|(f\circ\gamma)'(t)| &= \left|\lim_{h\to 0}\frac{(f\circ\gamma)(t+h)-(f\circ\gamma)(t)}{h}\right| \\
&= \lim_{h\to 0}\left|\frac{\sum_{i\in M_{i_0}} F_{i-1}\omega_i(\gamma(t+h)) - \sum_{i\in M_{i_0}} F_{i-1}\omega_i(\gamma(t))}{h}\right| \\
&= \lim_{h\to 0}\left|\frac{\sum_{i\in M_{i_0}} F_{i-1}\omega_i(\gamma(t+h)) - F_{i_0} + F_{i_0} - \sum_{i\in M_{i_0}} F_{i-1}\omega_i(\gamma(t))}{h}\right| \\
&= \lim_{h\to 0}\left|\frac{\sum_{i\in M_{i_0}} F_{i-1}\omega_i(\gamma(t+h)) - F_{i_0}\sum_{i\in M_{i_0}}\omega_i(\gamma(t+h))}{h}\right. \\
&\qquad \left. + \frac{F_{i_0}\sum_{i\in M_{i_0}}\omega_i(\gamma(t)) - \sum_{i\in M_{i_0}} F_{i-1}\omega_i(\gamma(t))}{h}\right| \\
&= \lim_{h\to 0}\left|\frac{\sum_{i\in M_{i_0}}(F_{i-1}-F_{i_0})\omega_i(\gamma(t+h)) - \sum_{i\in M_{i_0}}(F_{i-1}-F_{i_0})\omega_i(\gamma(t))}{h}\right| \\
&= \lim_{h\to 0}\left|\frac{\sum_{i\in M_{i_0}}(F_{i-1}-F_{i_0})(\omega_i(\gamma(t+h)) - \omega_i(\gamma(t)))}{h}\right| \\
&\le \lim_{h\to 0}\frac{\sum_{i\in M_{i_0}}|(F_{i-1}-F_{i_0})(\omega_i(\gamma(t+h)) - \omega_i(\gamma(t)))|}{|h|} \\
&\le \lim_{h\to 0}\frac{\sum_{i\in M_{i_0}}|F_{i-1}-F_{i_0}|\frac{A}{\varrho_i}|h|}{|h|} = \sum_{i\in M_{i_0}}|(F_{i-1}-F_{i_0})|\frac{A}{\varrho_i}.
\end{aligned}$$

As several times before, we prop upon the fact that

$$\sum_{i=1}^{\infty}\omega_i \equiv 1 \qquad (\text{ here } \sum_{j\in M_{i_0}}\omega_j(\gamma(t)) = 1)$$

and upon observation (3.20).

Hence,

$$
\begin{aligned}
&|(f\circ\gamma)'(t)|\\
&\le A\sum_{i\in M_{i_0}}\frac{|F_{i-1}-F_{i_0}|}{\varrho_i}\sum_{j\in M_{i_0}}\omega_j(\gamma(t))\\
&= A\sum_{j\in M_{i_0}}\sum_{i\in M_{i_0}}\frac{|F_{i-1}-F_{i_0}|}{\varrho_i}\omega_j(\gamma(t))\\
&= A\sum_{j\in M_{i_0}}\sum_{i\in M_{i_0}}\frac{|F_{i-1}-F_{j-1}+F_{j-1}-F_{i_0}|}{\varrho_i}\omega_j(\gamma(t))\\
&\le A^{2N+1}\sum_{j\in M_{i_0}}\sum_{i\in M_{i_0}}\frac{|F_{i-1}-F_{j-1}|+|F_{j-1}-F_{i_0}|}{\varrho_j}\omega_j(\gamma(t))\\
&= A^{2N+1}\sum_{j\in M_{i_0}}\left(\left(\sum_{i\in M_{i_0}}\frac{|F_{i-1}-F_{j-1}|}{\varrho_j}\right)\omega_j(\gamma(t))+\left(\sum_{i\in M_{i_0}}\frac{|F_{i_0}-F_{j-1}|}{\varrho_j}\right)\omega_j(\gamma(t))\right)\\
&= A^{2N+1}\sum_{j\in M_{i_0}}\left(\left(\sum_{i\in M_{i_0}}\frac{|F_{i-1}-F_{j-1}|}{\varrho_j}\right)\omega_j(\gamma(t))+(2N+1)\frac{|F_{i_0}-F_{j-1}|}{\varrho_j}\omega_j(\gamma(t))\right),
\end{aligned}
$$

everything again λ-a.e. on $(0,l(\gamma))$. Using that $(2N+1)\frac{|F_{i-1}-F_{j-1}|}{\varrho_j}\ge 0$ for the second term in outer sum, we conclude that λ-a.e. on $(0,l(\gamma))$

$$
\begin{aligned}
&|(f\circ\gamma)'(t)|\\
&\le A^{2N+1}\sum_{j\in M_{i_0}}\left(\left(\sum_{i\in M_{i_0}}\frac{|F_{i-1}-F_{j-1}|}{\varrho_j}\right)\omega_j(\gamma(t))+\left(\sum_{i\in M_{i_0}}(2N+1)\frac{|F_{i-1}-F_{j-1}|}{\varrho_j}\right)\omega_j(\gamma(t))\right)\\
&= A^{2N+1}\sum_{j\in M_{i_0}}\left(\left(\sum_{i\in M_{i_0}}(2N+2)\frac{|F_{i-1}-F_{j-1}|}{\varrho_j}\right)\omega_j(\gamma(t))\right)\\
&= A^{2N+1}(2N+2)\sum_{i\in M_{i_0}}\left(\sum_{j\in M_{i_0}}\frac{|F_{i-1}-F_{j-1}|}{\varrho_j}\omega_j(\gamma(t))\right)\\
&= A^{2N+1}(2N+2)\sum_{i\in M_{i_0}}\left(\sum_{j\in M_{i_0}}\frac{|F_{i-1}-F_{j-1}|}{\varrho_j}\omega_j(\gamma(t))\right)\chi_{\bigcup_{l\in M_i}\Omega_l}(\gamma(t))\\
&\le A^{2N+1}(2N+2)\sum_{i\in Q_{i_0}}\left(\sum_{j\in M_{i_0}}\frac{|F_{i-1}-F_{j-1}|}{\varrho_j}\omega_j(\gamma(t))\right)\chi_{\bigcup_{l\in M_i}\Omega_l}(\gamma(t))=(g\circ\gamma)(t).
\end{aligned}
$$

Thereby we have finished the proof that f belongs to the class of functions considered in statement (I) of the theorem.

Now, when we know that f satisfies the inequality in (I), the last step is to use it for appropriate estimates leading to establishing the inequality in (II) for function F. Since the function $F(t)$ is linear on each $[t_{i-1},t_i]$, so is the function $F(t)-a$ for any $a\in\mathbb{R}$. Therefore either $|F(t)-a|$ and then consequently also $|F(t)-a|^p$ is monotone on $[t_{i-1},t_i]$, or there exists a point $\xi_i\in[t_{i-1},t_i]$ for which $|F(\xi_i)-a|=0$ and $|F(t)-a|$

is nonincreasing on $[t_{i-1}, \xi_i]$ and nondecreasing on $[\xi_i, t_i]$ (again consequently so is also $|F(t) - a|^p$).

In the former case we have

$$\int_{t_{i-1}}^{t_i} |F(t) - a|^p dt \le \max\{|F_{i-1} - a|^p, |F_i - a|^p\}\varrho_i \le \big(|F_{i-1} - a|^p + |F_i - a|^p\big)\varrho_i,$$

while in the latter one, we have

$$\int_{t_{i-1}}^{t_i} |F(t)-a|^p dt = \int_{t_{i-1}}^{\xi_i} |F(t)-a|^p dt + \int_{\xi_i}^{t_i} |F(t)-a|^p dt \le \big(|F_{i-1}-a|^p + |F_i-a|^p\big)\varrho_i.$$

At any rate, we get

$$\begin{aligned}
&\|F - a\|^p_{L^p([0,b),d\nu)} \\
&\qquad = \int_0^b |F(t) - a|^p \varrho(t)^{n-1}\, dt = \sum_{i=1}^{\infty} \int_{t_{i-1}}^{t_i} |F(t) - a|^p \varrho(t)^{n-1}\, dt \\
&\qquad = \sum_{i=1}^{\infty} \int_{t_{i-1}}^{t_i} |F(t) - a|^p \varrho_i^{n-1}\, dt \le \sum_{i=1}^{\infty} \big(|F_i - a|^p + |F_{i-1} - a|^p\big)\varrho_i^n \\
&\qquad \le \sum_{i=1}^{\infty} \big(|F_i - a| + |F_{i-1} - a|\big)^p \varrho_i^n.
\end{aligned}$$

The condition (3.8) implies

$$⨍_{\Omega_i} \sum_{j \in M_i} \omega_j(x)\, d\boldsymbol{m}(x) = ⨍_{\Omega_i} \sum_{j=1}^{\infty} \omega_j(x)\, d\boldsymbol{m}(x) = 1.$$

Hence,

$$
\begin{aligned}
\|F-a\|^p_{L^p([0,b),d\nu)} &\le \sum_{i=1}^{\infty}\Big(\big|(F_i-a)\fint_{\Omega_i}\sum_{j\in M_i}\omega_j(x)\,d\boldsymbol{m}(x)\big| \\
&\quad + \big|(F_{i-1}-a)\fint_{\Omega_i}\sum_{j\in M_i}\omega_j(x)\,d\boldsymbol{m}(x)\big|\Big)^p\varrho_i^n \\
&= \sum_{i=1}^{\infty}\Big(\big|\fint_{\Omega_i}\sum_{j\in M_i}(F_i-a)\omega_j(x)\,d\boldsymbol{m}(x)\big| + \big|\fint_{\Omega_i}\sum_{j\in M_i}(F_{i-1}-a)\omega_j(x)\,d\boldsymbol{m}(x)\big|\Big)^p\varrho_i^n \\
&= \sum_{i=1}^{\infty}\Big(\big|\fint_{\Omega_i}\sum_{j\in M_i}(F_i-F_{j-1}+F_{j-1}-a)\omega_j(x)\,d\boldsymbol{m}(x)\big| \\
&\quad + \big|\fint_{\Omega_i}\sum_{j\in M_i}(F_{i-1}-F_{j-1}+F_{j-1}-a)\omega_j(x)\,d\boldsymbol{m}(x)\big|\Big)^p\varrho_i^n \\
&= \sum_{i=1}^{\infty}\Big(\big|\fint_{\Omega_i}\Big(\sum_{j\in M_i}(F_i-F_{j-1})\omega_j(x)+\sum_{j\in M_i}(F_{j-1}-a)\omega_j(x)\Big)\,d\boldsymbol{m}(x)\big| \\
&\quad + \big|\fint_{\Omega_i}\Big(\sum_{j\in M_i}(F_{i-1}-F_{j-1})\omega_j(x)+\sum_{j\in M_i}(F_{j-1}-a)\omega_j(x)\Big)\,d\boldsymbol{m}(x)\big|\Big)^p\varrho_i^n \\
&\le \sum_{i=1}^{\infty}\Big(\fint_{\Omega_i}\sum_{j\in M_i}|F_i-F_{j-1}|\omega_j(x)\,d\boldsymbol{m}(x)+\fint_{\Omega_i}\big|\sum_{j\in M_i}(F_{j-1}-a)\omega_j(x)\big|\,d\boldsymbol{m}(x) \\
&\quad + \fint_{\Omega_i}\sum_{j\in M_i}|F_{i-1}-F_{j-1}|\omega_j(x)\,d\boldsymbol{m}(x)+\fint_{\Omega_i}\big|\sum_{j\in M_i}(F_{j-1}-a)\omega_j(x)\big|\,d\boldsymbol{m}(x)\Big)^p\varrho_i^n \\
&\le \sum_{i=1}^{\infty}\Big(\sum_{k\in M_i}\fint_{\Omega_i}\sum_{j\in M_i}|F_{k-1}-F_{j-1}|\omega_j(x)\,d\boldsymbol{m}(x)+2\fint_{\Omega_i}\big|\sum_{j\in M_i}(F_{j-1}-a)\omega_j(x)\big|\,d\boldsymbol{m}(x)\Big)^p\varrho_i^n.
\end{aligned}
$$

We extract the inverse value of the measure of Ω_i in front of the parentheses, apply (3.2) on ϱ_i and (3.14) on terms of outer sum.
Whence,

$$
\begin{aligned}
&\|F-a\|^p_{L^p([0,b),d\nu)} \\
&\qquad \le 2^{p-1}\sum_{i=1}^{\infty}A\,\boldsymbol{m}(\Omega_i)^{1-p}\Big(\Big(\int_{\Omega_i}\sum_{k\in M_i}\sum_{j\in M_i}|F_{k-1}-F_{j-1}|\omega_j(x)\,d\boldsymbol{m}(x)\Big)^p \\
&\qquad\quad + \Big(2\int_{\Omega_i}\big|\sum_{j\in M_i}(F_{j-1}-a)\omega_j(x)\big|\,d\boldsymbol{m}(x)\Big)^p\Big) \\
&\qquad = 2^{p-1}A\sum_{i=1}^{\infty}\boldsymbol{m}(\Omega_i)^{1-p}\Big(\int_{\Omega_i}\sum_{k\in M_i}\sum_{j\in M_i}|F_{k-1}-F_{j-1}|\omega_j(x)\,d\boldsymbol{m}(x)\Big)^p \\
&\qquad\quad + 2^{2p-1}A\sum_{i=1}^{\infty}\boldsymbol{m}(\Omega_i)^{1-p}\Big(\int_{\Omega_i}\big|\sum_{j\in M_i}(F_{j-1}-a)\omega_j(x)\big|\,d\boldsymbol{m}(x)\Big)^p.
\end{aligned}
$$

By Hölder's inequality ,

$$\Big(\int_{\Omega_i}\sum_{k\in M_i}\sum_{j\in M_i}|F_{k-1}-F_{j-1}|\omega_j(x)\,d\boldsymbol{m}(x)\Big)^p\le \boldsymbol{m}(\Omega_i)^{p-1}\int_{\Omega_i}\Big(\sum_{k\in M_i}\sum_{j\in M_i}|F_{k-1}-F_{j-1}|\omega_j(x)\Big)^p\,d\boldsymbol{m}(x),$$

as well as

$$\Big(\int_{\Omega_i}\Big|\sum_{j\in M_i}(F_{j-1}-a)\omega_j(x)\Big|\,d\boldsymbol{m}(x)\Big)^p\le \boldsymbol{m}(\Omega_i)^{p-1}\int_{\Omega_i}\Big|\sum_{j\in M_i}(F_{j-1}-a)\omega_j(x)\Big|^p\,d\boldsymbol{m}(x).$$

Therefore

$$\begin{aligned}\|F-a\|^p_{L^p([0,b),d\nu)}\le 2^{p-1}A\sum_{i=1}^{\infty}\int_{\Omega_i}\Big(\sum_{k\in M_i}\sum_{j\in M_i}|F_{k-1}-F_{j-1}|\omega_j(x)\Big)^p\,d\boldsymbol{m}(x)\\ +2^{2p-1}A\sum_{i=1}^{\infty}\int_{\Omega_i}\Big|\sum_{j\in M_i}(F_{j-1}-a)\omega_j(x)\Big|^p\,d\boldsymbol{m}(x).\end{aligned}$$

Considering (3.19), (3.2) and (3.8) we find the proper upper bounds of both summands, namely,

$$\begin{aligned}&\sum_{i=1}^{\infty}\int_{\Omega_i}\Big(\sum_{k\in M_i}\sum_{j\in M_i}|F_{k-1}-F_{j-1}|\omega_j(x)\Big)^p\,d\boldsymbol{m}(x)\\
&\quad=\sum_{i=1}^{\infty}\int_{\Omega_i}\Big(\sum_{k\in M_i}\sum_{j\in M_i}\frac{|F_{k-1}-F_{j-1}|}{\varrho_j}\varrho_j\omega_j(x)\Big)^p\,d\boldsymbol{m}(x)\\
&\quad\le\sum_{i=1}^{\infty}\int_{\Omega_i}\Big(\sum_{k\in M_i}\sum_{j\in M_i}\frac{|F_{k-1}-F_{j-1}|}{\varrho_j}A^N\varrho_i\omega_j(x)\Big)^p\,d\boldsymbol{m}(x)\\
&\quad\le A^{Np}\sum_{i=1}^{\infty}(A\,\boldsymbol{m}(\Omega_i))^{\frac{p}{n}}\int_{\Omega_i}\Big(\sum_{k\in M_i}\sum_{j\in M_i}\frac{|F_{k-1}-F_{j-1}|}{\varrho_j}\omega_j(x)\Big)^p\,d\boldsymbol{m}(x)\\
&\quad= A^{Np}\sum_{i=1}^{\infty}(A\,\boldsymbol{m}(\Omega_i))^{\frac{p}{n}}\int_{\Omega_i}\Big(\sum_{k\in M_i}\Big(\sum_{j\in M_i}\frac{|F_{k-1}-F_{j-1}|}{\varrho_j}\omega_j(x)\Big)\chi_{\bigcup\limits_{l\in M_k}\Omega_l}(x)\Big)^p\,d\boldsymbol{m}(x)\\
&\quad\le A^{Np+\frac{p}{n}}(\boldsymbol{m}(\Omega))^{\frac{p}{n}}\sum_{i=1}^{\infty}\int_{\Omega_i}\Big(\sum_{k=1}^{\infty}\Big(\sum_{j=1}^{\infty}\frac{|F_{k-1}-F_{j-1}|}{\varrho_j}\omega_j(x)\Big)\chi_{\bigcup\limits_{l\in M_k}\Omega_l}(x)\Big)^p\,d\boldsymbol{m}(x)\\
&\quad\le A^{Np+\frac{p}{n}}(\boldsymbol{m}(\Omega))^{\frac{p}{n}}(2N+1)\int_{\Omega}\Big(\sum_{k=1}^{\infty}\Big(\sum_{j=1}^{\infty}\frac{|F_{k-1}-F_{j-1}|}{\varrho_j}\omega_j(x)\Big)\chi_{\bigcup\limits_{l\in M_k}\Omega_l}(x)\Big)^p\,d\boldsymbol{m}(x)\\
&\quad= C(A,N,n,p,\Omega)\|g\|^p_{L^p(\Omega,d\boldsymbol{m})}\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{\infty}\int_{\Omega_i}\Big|\sum_{j\in M_i}(F_{j-1}-a)\omega_j(x)\Big|^p\,d\boldsymbol{m}(x) &= \sum_{i=1}^{\infty}\int_{\Omega_i}\Big|\sum_{j\in M_i}F_{j-1}\omega_j(x)-a\sum_{j\in M_i}\omega_j(x)\Big|^p\,d\boldsymbol{m}(x)\\ &= \sum_{i=1}^{\infty}\int_{\Omega_i}\Big|\sum_{j=1}^{\infty}F_{j-1}\omega_j(x)-a\Big|^p\,d\boldsymbol{m}(x)\\ &\leq (2N+1)\int_{\Omega}\Big|\sum_{j=1}^{\infty}F_{j-1}\omega_j(x)-a\Big|^p\,d\boldsymbol{m}(x)\\ &= (2N+1)\int_{\Omega}\big|f(x)-a\big|^p\,d\boldsymbol{m}(x)\\ &= C(N)\|f-a\|^p_{L^p(\Omega,d\boldsymbol{m})}.\end{aligned}$$

To conclude, a successive use of the inequality from (I) with f and (3.21) shows that

$$\begin{aligned}\inf_{a\in\mathbb{R}}\|F-a\|^p_{L^p([0,b),d\nu)} &\leq \inf_{a\in\mathbb{R}}\Big(C(A,N,n,p,\Omega)\|g\|^p_{L^p(\Omega,d\boldsymbol{m})}+C(A,N,p)\|f-a\|^p_{L^p(\Omega,d\boldsymbol{m})}\Big)\\ &\leq C(A,N,n,p,\Omega)\|g\|^p_{L^p(\Omega,d\boldsymbol{m})}+C(A,N,p)\inf_{a\in\mathbb{R}}\|f-a\|^p_{L^p(\Omega,d\boldsymbol{m})}\\ &\leq C(A,N,n,p,\Omega)\|g\|^p_{L^p(\Omega,d\boldsymbol{m})}\\ &\leq C(A,N,n,p,\Omega)\|F'\|^p_{L^p([0,b),d\nu)},\end{aligned}$$

as desired.

$\square$

Bibliography

[1] Adams, R. A.: *Sobolev Spaces*, Academic Press Inc., New York, 1975.

[2] Evans, W. D. and Harris, D. J.: *Sobolev embeddings for generalized ridged domains*, Proc. London Math. Soc. (2) **54** (1987), 141-175.

[3] Hajłasz, P.: *Sobolev spaces on metric-measure spaces*, Contemp. Math. **338** (2003).

[4] Hajłasz, P. and Koskela, P.: *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145** (2000), no. 688, x+101 pp.

[5] Heinonen, J.: *Lectures on Analysis on Metric Spaces*, Springer-Verlag Inc., New York, 2001.